Coherence via big categories with families of locally cartesian closed categories

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Locally cartesian closed (lcc) categories are natural categorical models of dependent type theory [See84]. However, there is a slight mismatch: syntactic substitution is functorial and commutes strictly with type formers, whereas pullback is generally only pseudo-functorial and preserves universal objects only up to isomorphism. In response to this problem, several notions of models with strict pullback operations have been introduced, e.g. categories with families (cwf) [Dyb96], and coherence techniques have been developed to *strictify* weak models such as lcc categories and obtain models with functorial substitution [CGH14][LW15]. While an interpretation of almost all of homotopy type theory is known to exist in arbitrary infinity toposes [Shu19], even an interpretation of just intensional dependent type theory in arbitrary lcc quasi-categories is only conjectured [KS17].

This talk introduces the big cwfs of lcc 1-categories and lcc quasi-categories, a novel coherence construction for dependent type theories. In the 1-categorical case, the technique provides a new proof of the interpretation of extensional type theory in lcc categories. The big cwf of lcc quasi-categories supports substitution-stable type and term formers for dependent sums and products, but expected judgemental laws such as β equality hold only up to contractible path equality. I expect that this structure suffices to interpret a hypothetical weak variant of intensional type theory.

Our point of departure is the observation that, when working in type theory, changing the ambient context is akin to changing the base terms of the underlying theory. For example, proving $v : \sigma \vdash t : \tau$ is equivalent to proving $\cdot \vdash t : \tau$ in a type theory that was freely extended by a term v of type σ. We take the idea that contexts represent different type theories literally and assign to each context a separate model, i.e. a separate lcc category. Context extension then corresponds to freely adjoining an interpretation of a term to an lcc category.

The big cwf of lcc quasi-categories is defined analogously to the big cwf of lcc 1-categories; the definition of the former is thus omitted here for brevity.

Definition 1. The *big cwf of lcc categories* is given as follows.

- A context is an lcc category Γ equipped with equivalences
$$\mathrm{Lcc}(\Gamma, \mathcal{C}) \simeq \mathrm{sLcc}(\Gamma, \mathcal{C})$$
for all lcc categories $\mathcal{C}$, 2-natural wrt. strict lcc functors in $\mathcal{C}$.
- A type in context Γ is an object $\sigma \in \mathrm{Ob}\,\Gamma$.
- A term of type σ is a morphism $1 \to \sigma$ in Γ.
- A context morphism from Γ to Δ is a strict lcc functor $\Delta \to \Gamma$.
- The extended context $\Gamma.\sigma$ is obtained by freely adjoining a morphism $1 \to \sigma$ to Γ.

*Supported by AFOSR grant 12595060.

Theorem 1. *The big cwf of lcc 1-categories contains every lcc category up to equivalence, it has an initial context and comprehensions, and it supports* Σ, Π *and extensional* Eq *types.*

Theorem 1 enables the interpretation of a type theory with the corresponding features in (slices of) the big cwf. In the 1-categorical case, the existence of a context $\Gamma_{\mathcal{C}} \simeq \mathcal{C}$ equivalent to a given lcc category $\mathcal{C}$ follows from the 2-monadicity of the category of strict lcc categories over the $(2,1)$-category of categories [BKP89]. Roughly speaking, $\Gamma_{\mathcal{C}}$ is obtained from $\mathcal{C}$ by forgetting about the canonical choice of lcc structure and adjoining a new one. $\Gamma_{\mathcal{C}}$ can thus be seen as a *destrictification* of $\mathcal{C}$. For example, even when $\mathcal{C}$ happens to be equipped with a strictly functorial choice of pullback functors, the canonical pullback functors of $\Gamma_{\mathcal{C}}$ will not have this property.

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