WIP: Coherence via big categories with families of locally cartesian closed categories

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The coherence problem

Locally cartesian closed (lcc) categories are natural categorical models of dependent type theory.

Substitution	Pullback
strictly functorial	functorial up to iso
$\tau[s_2][s_1] = \tau[s_2[s_1], s_1]$	$s_1^st(s_2^st(au))\cong(s_2\circ s_1)^st(au)$
commutes with type formers	preserves structure up to iso
$(au_1 ightarrow au_2)[s] = au_1[s] ightarrow au_2[s]$	$s^*(au_2^{ au_1})\cong s^*(au_2)^{s^*(au_1)}$

\implies Cannot interpret syntax directly

Coherence constructions

Prior art:

- Curien Substitution up to isomorphism
- Hofmann Giraud-Bénabou construction
- Lumsdaine, Warren, Voevodsky (local) universes

These constructions interpret type theory in a given single lcc category.

This talk: Interpret type theory in the ("gros") category of all lcc categories.

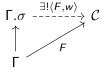
- Interpretation of extensional type theory in a single lcc 1-category can be recovered by slicing
- Expected to interpret a (as of now, hypothetical) weak variant of intensional dependent type theory in arbitrary lcc quasi-categories

The big cwfs of lcc categories

Definition

Let $r \in \{1, \infty\}$. The cwf **Lcc**_r is given as follows:

- A context is a cofibrant lcc r-category Γ.
- Ty(Γ) = Ob Γ
- $\blacktriangleright \operatorname{Tm}(\Gamma, \sigma) = \operatorname{Hom}_{\Gamma}(1_{\Gamma}, \sigma)$
- $\blacktriangleright \operatorname{Hom}_{\operatorname{Ctx}}(\Gamma, \Delta) = \operatorname{Hom}_{\operatorname{sLcc}}(\Delta, \Gamma)$
- F.σ is obtained from Γ by adjoining freely a morphism v : 1 → σ:



with F strict, $w: 1 \to F(\sigma)$ in C and $\langle F, w \rangle(v) = w$.

Main results

Theorem Let $r \in \{1, \infty\}$.

- The functors Lcc^{op}_r → Lcc_r are equivalences of (2,1) resp. (∞,1) categories.
- Context extension in Lcc_r is well-defined.
- Denote by (Lcc_r)_{*} the category of pairs (Γ, σ) of contexts equipped with a base type σ ∈ Ty(Γ). Then the two functors (Lcc_r)^{op}_{*} → Lcc_r

$$(\Gamma, \sigma) \mapsto \Gamma. \sigma$$
 $(\Gamma, \sigma) \mapsto \Gamma_{/\sigma}$

are strictly naturally equivalent.

Lcc₁ supports Π , Σ , (extensional) Eq and Unit types.

Recovering an interpretation in a single lcc category

Corollary

Every lcc 1-category C is equivalent to a cwf supporting Π , Σ , (extensional) Eq and Unit types.

Proof.

Let $\Gamma_{\mathcal{C}} \in Lcc$ such that $\Gamma_{\mathcal{C}} \simeq \mathcal{C}$ as lcc categories. Let C be the least full on types and terms sub-cwf of $Lcc_{/\Gamma_{\mathcal{C}}}$ supporting the type constructors above. Then $C \simeq \Gamma_{\mathcal{C}} \simeq \mathcal{C}$ as lcc categories.

J-algebras

Definition

Let J be a set of morphisms in a category C. A J-algebra is an object X of C equipped with lifts



for all $j \in J$ and arbitrary p. A *J*-algebra morphism is a morphism in C compatible with the $\ell_j(p)$. The category of *J*-algebras is denoted by $\mathcal{A}(J)$.

Duality of structure and property

Theorem (Nikolaus 2011)

Let \mathcal{M} be a cofibrantly generated locally presentable model category whose cofibrations are the monomorphisms. Let J be a set of trivial cofibrations such that an object (!) X is fibrant iff it has the rlp. wrt. J. Denote by $\mathcal{A} = \mathcal{A}(J)$ the category of J-algebras. Then the evident forgetful functor $R : \mathcal{A} \rightleftharpoons \mathcal{M} : L$ is a right adjoint (even monadic). The model category structure of \mathcal{M} can be transferred to \mathcal{A} , and (R, L) is a Quillen equivalence.

in ${\cal M}$	in ${\cal A}$
all objects are cofibrant	all objects are fibrant
codomains might not have enough properties	domains might be too structured
$X \to R(L(X))$ is fib. replacement	L(R(Y)) ightarrow Y is cof. replacement

Model categories of lcc categories

Assumption

Let $r \in \{1, \infty\}$. There are cofibrantly generated locally presentable model categories Lcc_r such that

- the cofibrations are the monomorphisms,
- the fibrant objects are lcc 1-categories resp. lcc quasi-categories, and
- the weak equivalences of fibrant objects are equivalences of (quasi-)categories.

Definition

Fix sets $J_r \subseteq \operatorname{Lcc}_r$ as in Nikolaus's theorem. The category of strict lcc *r*-categories is given by $\operatorname{sLcc}_r = \mathcal{A}(J_r)$.

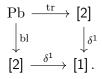
Thus $(\mathbf{Lcc}_r)^{\mathrm{op}} \subseteq \mathrm{sLcc}_r$ is the full subcategory of cofibrant objects.

Marking universal objects

Idea to construct Lcc_r : A category of (separated) presheaves over some base category S containing objects corresponding to universal objects.

Example

 $(S_{
m Pb})^{
m op}$ is generated by $\Delta
ightarrow S_{
m Pb}$ and a commuting square



Then $\mathcal{M} = \{X \in \widehat{S_{\mathrm{Pb}}} \mid \mathrm{tr}, \mathrm{bl} : X_{\mathrm{Pb}} \rightrightarrows X_2 \text{ is jointly mono}\} \text{ and } J_{\mathrm{Pb}}$ is chosen such that $(\Lambda_k^n \subset \Delta^n) \in J_{\mathrm{Pb}}$ and

- marked squares have the universal property of pullback squares;
- there is a marked square completing any given cospan;
- marked squares are closed under isomorphism.

Proofs of the main results

Proposition

The functors $(Lcc_r)^{\mathrm{op}} \to Lcc_r$ are equivalences of (2,1) resp. $(\infty, 1)$ categories.

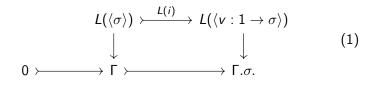
Proof.

 ${\rm sLcc}_r\to {\rm Lcc}_r$ is an equivalence by Nikolaus's theorem and $(\textbf{Lcc}_r)^{\rm op}={\rm sLcc}_{\rm cof}.$

Proposition

Context extension in Lcc_r is well-defined.

Proof.



Proposition

Denote by $(\mathbf{Lcc}_r)_*$ the category of pairs (Γ, σ) of contexts Γ equipped with a base type $\sigma \in \mathrm{Ty}(\Gamma)$. Then the two functors $(\mathbf{Lcc}_r)^{\mathrm{op}}_* \to \mathrm{Lcc}_r$

$$(\Gamma, \sigma) \mapsto \Gamma. \sigma \qquad \qquad (\Gamma, \sigma) \mapsto \Gamma_{/\sigma}$$

are strictly naturally equivalent.

Proof.

 $\Gamma_{/\sigma}$ is a homotopy pushout of (1) in Lcc_r .

Proposition

Lcc₁ supports Π , Σ , (extensional) Eq and Unit types.

Proof $(\mathbf{\Pi})$.

Suppose $\Gamma . \sigma \vdash \tau$. Define $\Gamma \vdash \mathbf{\Pi}_{\sigma}(\tau)$ as image of τ under

$$\Gamma.\sigma \xrightarrow{D} \Gamma_{/\sigma} \xrightarrow{\Pi_{\sigma}} \Gamma$$

Now suppose $\Gamma \vdash u : \Pi_{\sigma} \tau$. Then $\tilde{u} : \sigma^*(1) \to D(\tau)$ in $\Gamma_{/\sigma}$ by transposing along $\sigma^* \dashv \Pi_{\sigma}$. Map \tilde{u} via $E : \Gamma_{/\sigma} \xrightarrow{\sim} \Gamma.\sigma$ and compose with component of natural equivalence $E(D(\tau)) \simeq \tau$ to obtain $\Gamma.\sigma \vdash \operatorname{App}(u) : \tau$.

For $r=\infty$ all non-trivial equalities hold only up to path equality, e.g. the β law

$$\operatorname{App}(\lambda u)) = u$$

holds only up to a contractible choice of path.

Future work

Can some kind of weak type theory be interpreted in $Lcc_{\infty}?$

The unit $\mathcal{C} \to \mathcal{C}^s$ arising from freely turning a monoidal category \mathcal{C} into a *strict* monoidal category \mathcal{C}^s is not generally an equivalence, but MacLane's theorem shows that it is if \mathcal{C} is cofibrant. Does this also work with lcc quasi-categories and a notion of strictness where e.g. the canonical simplex corresponding to β is required to be a degenerate?

Conclusion

- Yet another solution to the coherence problem for extensional dependent type theory
- Interpret in category of all lcc categories instead of a single one, recover interpretation in a single lcc by slicing
- Restrict to cofibrant objects in algebraic presentation of model categories of lcc categories
- Model context extension as 1-categorical pushout, not slice category
- Solves at least pullback coherence for quasi-categories
- Stephen Lack, *Homotopy-theoretic aspects of 2-monads.*, Journal of Homotopy & Related Structures **2** (2007), no. 1.
- Thomas Nikolaus, *Algebraic models for higher categories*, Indagationes Mathematicae **21** (2011), no. 1-2, 52–75.