Master Thesis

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Categories with algebraic structure as models of partial Horn logic theories

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Summary

We show how the categories of models for partial Horn logic theories can be described as orthogonal subcategories of a category of partial algebras. Free models arising from theory morphisms can then be obtained by the usual smallness argument via an inductive construction.

A factorization system of totalizing and total morphisms is used to define the validity checking problem. It asks whether a given syntax tree over a signature and given base terms is well-formed in the free model of the respective partial Horn logic theory. Validity checking can be understood as a weak form of type checking in the setting of an arbitrary partial Horn logic theory. We show that, under mild hypothesis, it is equivalent to deciding equality in free models.

Various examples for partial Horn logic theories are given, namely for categories, left exact (= finitely complete) categories, locally cartesian closed (lcc) categories and elementary toposes. The categories arising from the model structure are usually too rigid for most applications; for example, model morphisms of left exact categories have to preserve choices of canonical limits on the nose and not just up to isomorphism. To each of the above-mentioned types of algebraic structure on categories, we define a corresponding theory of sketches, for example the theory of lcc sketches. Using a simple bicategorical argument based on the preservation of strong inserters, we prove in each case that firstly the 1-categorical adjunction between the models of the category theory and the models of the sketch theory extends to a 2-adjunction, and that secondly this 2-adjunction induces a biadjunction between the category of structure up to isomorphism preserving functors and the category of sketches.

As a special case, we obtain a syntactic description of the bifree lcc category over a single object without type theoretic arguments. A proof for the undecidability of equality in the bifree lcc category over a single object is translated from the type theoretical setting and proved directly. It follows that this instance of the validity checking problem is undecidable.

Contents

1 Introduction

Martin-Löf (ML) type theory [11] is a framework for formal constructive reasoning and constructive mathematics. It was used successfully to establish computer-verified proofs in various mathematical domains [10] [9]. While initially not intended to serve as such, ML type theory is also the *internal language* for locally cartesian closed (lcc) categories.

What is usually meant by this are the following facts. ML type theory can, essentially by definition, be interpreted in every category with families (cwf) [22] [6]. From a categorical point of view, this interpretation is an initiality statement: The syntax of ML type theory over given base terms modulo definitional equality can be assembled into a cwf, and this syntactic cwf C is the *free* cwf over the base terms. The "interpretation" is the unique morphism $C \to D$ into any given cwf over the same base terms.

However, initiality holds only among strict cwf morphisms, i.e. the ones that preserve all structure on the nose and not just up to isomorphism like the *pseudo-morphisms*. An analogous statement for pseudo-morphisms can be recovered if we consider the category of cwf and pseudo-morphisms as a 2-category: In [6], the free (among strict morphisms) cwf over a single base type is proved to be *bifree*, i.e. free up to isomorphism, among the pseudo-morphisms.

It has been "known" for some time that there is an equivalence between the category of cwf and the category of lcc categories [20]. However, this result had to be rectified [7] to a *biequivalence* between the 2-category of cwf and pseudo-morphisms and the 2-category of lcc categories and lcc functors, i.e functors which preserve all lcc structure up to isomorphism. Because biinitiality is preserved under biequivalences, the syntactic cwf C over a single base type is mapped to the (or rather, *a*) bifree lcc category $\mathcal{C}$ over a single object; the essentially unique morphism $\mathcal{C} \to \mathcal{D}$ to any other lcc category $\mathcal{D}$ over the base object is then the "interpretation" of ML type theory in the lcc category $\mathcal{D}$.

Let us demonstrate on a simple example how these results can be used to prove simple facts about lcc categories. Say we want to prove that for every object X in an lcc category $\mathcal{C}$ there is an isomorphism

$$(X \times X)^X \cong X^X \times X^X. \tag{1.1}$$

This isomorphism can be proved to exist in general cartesian closed categories via the simply typed lambda calculus, but the general scheme remains the same. We first reduce to the case of $\mathcal{C}$ being the bifree lcc category over X, because then we can transport the isomorphism (1.1) along the essentially unique lcc morphism into any other lcc category. Next, we can translate (1.1) into type theoretic language because of the biequivalence alluded to above and instead consider the syntactic cwf over a single base type X. Recall that products $X \times X$ are simply non-dependent sum types $\sum_X X$, and that likewise the non-dependent exponential X^X is given up to isomorphism by $\prod_X X$. Now to complete the proof for (1.1), it suffices to construct terms x', y' such that

$$x : (X \times X)^X \vdash x' : X^X \times X^X \qquad\qquad y : X^X \times X^X \vdash y' : (X \times X)^X$$

corresponding to the isomorphism and its supposed inverse and to prove

$$x : (X \times X)^X \vdash y'[y \coloneqq x'] \equiv x : X^X \times X^X$$

$$y : X^X \times X^X \vdash x'[x \coloneqq y'] \equiv y : (X \times X)^X,$$

showing that x' and y' are indeed mutually inverse. We also refer to the the introduction of the unpublished draft [21] for a more thorough explanation of this technique.

We sum up the statement that ML type theory is the internal language of lcc categories as *(i)* the syntax of ML type theory gives rise to bifree objects in the 2-category of cwf with pseudo-morphisms; and

(ii) there is a biequivalence between the 2-categories of cwf with pseudo-morphisms and the 2-category of lcc categories.

If instead of *extensional* Id types, *intensional* Id types are considered, there is an open internal language hypothesis that relates intensional ML type theory with universes and the univalence axiom to infinity toposes.[1][2]

In both cases, however, the statement that ML type theory is *the* internal language, as opposed to *a* internal language, is slightly imprecise, for clearly this notion is well-defined only up to the respective notions of equivalence (biequivalence or equivalence of infinity categories). Regarding ML type theory as just one possible internal language for lcc categories, it is not clear what purpose the 2-category of cwf has, i.e. why we have to find a 2-category different from the one we are ultimately interested in for a convenient syntax. Clearly it would be conceptually easier to simply forego the construction the biequivalence from point *(ii)* and instead find a syntactical presentation for free lcc categories directly.

Lcc categories are just one instance of categories with algebraic structure, which are objects of study of 2-monad theory [3] [18]. Other examples are cartesian closed categories (corresponding to simply typed lambda calculus) and symmetric monoidally closed categories (corresponding to linear logic/types), which can all be exhibited as algebras of certain 2-monads. The construction of bifree categories with algebraic structure follows from the generality of 2-monad theory. For example, we can consider the bifree lcc category over a category or the bifree cartesian closed category over a symmetric monoidally closed category.

Often, as in the example proof of the isomorphism (1.1), one is interested in free categories with algebraic structure over finite "blueprints" from which complete categories can be constructed freely. For example, we might want to consider a free lcc category over a pullback square

$$\begin{tikzcd} x \arrow[r, equal, "\mathrm{id}"] \arrow[d, equal, "\mathrm{id}"'] \arrow[dr, phantom, "\lrcorner", very near start] & x \arrow[d, "f"] \\ x \arrow[r, "f"'] & x \end{tikzcd} \tag{1.2}$$

(thus essentially over a single monomorphism $f : x \to x$). Indeed, there exists a bifree lcc category $\mathcal{C}$ over this finite datum, but $\mathcal{C}$ is not finite. Data such as (1.2) are known as *sketches* [13], in this case an lcc sketch (or just finite limit sketch because the closed structure does not occure). Usually, only finite (co)limit sketches are considered [2] [12], but in fact an appropriate notion of sketch is known for general 2-monads and free or bifree algebras of the respective 2-monad always exist, although sketches are usually assumed to come with underlying categories (whereas for the sketch (1.2), $f \circ f$ is not defined).

However, the way these results are usually obtained does not give rise to syntactical presentations of bifree categories with algebraic structure because they are usually proved with e.g. the (enriched) adjoint functor theorem. On the other hand, most programming language tools and proof assistants are syntax based. If one wants to use the canonical "language" of lcc categories, i.e. free lcc categories, as foundation of computer-based proof assistants, syntactic presentations of free lcc categories are needed. To bridge this gap, the author proposes the usage of *partial Horn logic* [17].

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[1] http://uwo.ca/math/faculty/kapulkin/seminars/hottestfiles/Shulman-2018-04-12-HoTTEST.pdf

[2] https://coq.inria.fr/files/coq5-slides-spitters.pdf

In [17], partial Horn logic theories whose models are precisely categories, left exact (= finitely complete) categories and locally cartesian closed categories, respectively, are constructed. From the free models arising from theory morphism, these theories provide concrete syntactical descriptions of free categories in these cases. There are various other logical systems with equivalent expressive power, for example cartesian theories [12] or essentially algebraic theories [2]. We agree with the authors of [17] however in that partial Horn logic is the most convenient for defining concrete theories (as opposed to studying models) and point out the striking similarity of partial Horn theories to the way type theory is usually presented (see section 4).

This work augments the results of [17] along the following axes.

(i) In all cases, the morphism arising from the description of the respective categories with algebraic structure as models of theories are *strict* functors, i.e. functors that preserve all relevant structure not only up to isomorphism but on the nose. Furthermore, the free models are a priori free only in a 1-categorical sense, i.e. arise from a 1-categorical adjunction instead of a 2-adjunction. Using a simple argument based on the preservation of a special type of 2-limit, the *strong inserter*, we deduce from the 1-categorical universal property a 2-universal one. From there, it is possible to prove that the respective free categories are also bifree if we relax the notion of morphism between them to the usual one, i.e. functors preserving structure up to isomorphism, using established results from 2-monad theory ([3], Theorem 5.1). However, this can also be done directly without much additional effort and is hence also shown here (lemma 4.2.4 and its applications).

(ii) We set up the respective theories with sketches in mind, so that the theory for the respective categories with algebraic structure is an extension of the corresponding theory for sketches. The 1-categorical adjunction arising from the theory extension is then proved to extend to a 2-adjunction or biadjunction.

(iii) In addition to the types of categories considered in [17], a theory whose models are precisely the (elementary) toposes is constructed.

The syntax of ML type theory is usually presented as a grammar generating *preterms and pretypes*, from which the well-formed ones are selected by a set of inference rules. The user of a proof assistant based on type theory produces a list of preterms, and the proof checker verifies whether these preterms are well-formed. (This is a simplified view of course; in practice, all proof assistants perform elaboration so that preterms can be specified more succinctly or are tactics based so that the user is barely confronted with preterms of the type theory at hand.)

In passing from type theory to a general partial Horn logic theory, an anologous concept is needed. Every partial Horn logic theory $\mathbb{T}$ is based on a (multisorted) signature Σ, and models of the theory are certain *partial* Σ*-algebras*. Here, the partiality refers to the partiality of the interpretation of operation symbols in partial algebras; they have to be interpreted as total functions in the usual (total) algebras. Partiality is necessary for the the kind of models we have in mind; consider as an example categories. An appropriate theory for categories (see also definition 4.3.1) will include sorts Mor and Ob for morphisms and an operation $\circ : \mathrm{Mor} \times \mathrm{Mor} \to \mathrm{Mor}$ for composition, but clearly $\circ$ cannot be total because it is defined only for morphisms with compatible source and target.

In [17], initial and then free models as arising from theory morphism are constructed using a set of inference rules. Initial models are then given by the partial algebra of all derivable terms modulo derivable equality. The approach presented here is more algebraic in nature: To each sequent in partial Horn logic, we assign an epimorphism of finite partial algebras. The models are then the partial algebras orthogonal to these epimorphisms, and free models are obtained using a variant of the *orthogonal-reflection construction* [2].

Our more algebraic approach has the disadvantage that it is at first not obvious how validity checking should be interpreted. To this end, we introduce the factorization system of *total* and *totalizing* morphisms. A total morphism $f : X \to Y$ of partial algebras can be interpreted as a (total) algebra relative to Y, in the sense that operations are defined on elements of X if and only if they are defined after mapping to Y. On the other hand, a totalizing morphism can be thought of as freely making operations more defined, i.e. adjoining syntax trees to a partial algebra. We can then factor a reflection $X \to X'$ into the category of models of a partial Horn logic theory as

$$X \to X_0 \to X'$$

with $X \to X_0$ totalizing and $X_0 \to X'$ an effective epimorphism. X_0 is a partial subalgebra of the totalization X^{tot} of X, i.e. the free algebra over X. Validity checking is now defined as decision problem for the inclusion $X_0 \to X^{\mathrm{tot}}$ (definition 3.4.2).

The crucial advantage of intensional ML type theory over the extensional variant is the fact that term equality (i.e. definitional equality) is decidable for the former using normalization-by-evaluation [1]. In extensional ML type theory, there are more definitional equalities, which makes term equality undecidable. In [6], the undecidability of terms in extensional ML type theory over a single base type is used to prove the undecidability of morphisms in the corresponding bifree lcc category, leaving a direct proof for future work. We adapt the proof given there for extensional ML type theory to lcc categories and hence obtain their result directly.

Throughout this paper, passing familiarity with category theory as can be obtained from e.g. [14] or [19] is assumed. Partial Horn logic being a central element in sections 3 and 4, the reader will find [17] helpful, although our exposition is self contained. The 2- or bi-categorical notions needed in section 4 are explained there, but some familiarity with bicategory theory is advantageous, e.g. from [12] or [8].

Given that we use our formalism to define and analyze decision problem, we do not make use of classical reasoning such as the law of the excluded middle or choice principles and instead rely solely on constructive reasoning [15]. In particular, all definitions and theorems allow for a computational interpretation via the effective topos [16]. We also assume the existence of a Grothendieck universe [23] in the meta logic to deal with size issues. Sets classified by the Grothendieck universe will be refered to as *small sets* while not necessarily small ones as *classes*. However, as assignment of set size is usually straightforward, these issues will be ignored most of the time. For example, in section 4, we define a 2-category Cat of categories, but technically, this is the 2-category of *small* categories, i.e. of categories $\mathcal{C}$ such that the set $\mathrm{Ob}\,\mathcal{C}$ of objects is small.

1.1 Summary of contributions

This work introduces the factorization system of totalizing and total morphisms (section 3.2) in the category of partial algebras. Using this factoriztion system, the construction of free models of partial Horn logic theories as provably defined terms modulo provable equality can be interpreted as factorization of a reflection of a partial algebra into the category of the theory's models. This allows for an algebraic interpretation and formalization of the *validity checking problem*, which is defined here and asks whether terms over a signature are well-defined in the models of a given partial Horn logic theory.

Partial Horn logic theories are essentially equivalent to sets of epimorphisms in categories of partial algebras (theorem 3.3.6). The proof presented here is novel, but the author learned after finishing this manuscript that the statement had already appeared in the work of B. Burmeister [5].

Although partial Horn logic theories for all the kinds of algebraic structure considered here except for the category of toposes have already appeared in [17], we give alternate theories modelling the universal properties of *all* universal objects of the respective kind explicitly instead of just those of the canonical universal objects. This makes the definition of sketches and and the construction of (bi)free categories with additional structure essentially straightforward.

Lastly, we give a direct proof for the main result of [6], where undecidability of a bifree lcc category over a single object is proved via an equivalent type theoretic statement. It follows that validity as defined here is also undecidable for the bifree lcc category over a single object.

2 Orthogonality

The free model theorem for partial Horn logic theories is established in [17] using a logical calculus, in particular without any advanced categorical concepts. In our exposition, the same theorem (and more) is proved algebraically using the general theory of *factorization systems*. This section introduces all relevant concepts and provides proofs for facts needed later on. The key result (theorem 2.2.7) can also be found in [4]. While the notions of M*-extending* and M*-collapsing* squares are novel, they are auxiliary in nature.

We work within a locally small category $\mathcal{C}$ throughout the section.

2.1 Orthogonal systems and factorization systems

Definition 2.1.1. Let $l : A \to B$ and $r : X \to Y$ be morphisms in $\mathcal{C}$. l is *orthogonal* to r, symbolically $l \perp r$, if for every commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "l"'] & X \arrow[d, "r"] \\ B \arrow[r] & Y \end{tikzcd}$$

there is a unique morphism $u : B \to X$ such that

$$\begin{tikzcd} A \arrow[r] \arrow[d, "l"'] & X \arrow[d, "r"] \\ B \arrow[r] \arrow[ur, dashed, "\exists! u"] & Y \end{tikzcd}$$

commutes. We also say that l is *left-orthogonal* to r and similarly that r is *right-orthogonal* to l. If $\mathcal{C}$ contains a terminal object $\top$, then an object Z is *orthogonal* (or *right-orthogonal*) to l if the unique morphism $X \to \top$ is right-orthogonal to l.

Let $M \subseteq \operatorname{Mor} \mathcal{C}$ be a class of morphisms in $\mathcal{C}$. We denote by

$$M^{\perp} = \{f \in \operatorname{Mor} \mathcal{C} \mid \forall c \in M : c \perp f\}$$

the class of morphisms orthogonal to every morphism in $c \in M$ and by

$$^{\perp}M = \{c \in \operatorname{Mor} \mathcal{C} \mid \forall f \in M : c \perp f\}$$

the class of morphism f such that every morphism in M is orthogonal to f. A class of morphisms is a *right-orthogonality class* if it is of the form $M^{\perp}$, and analogously a *left-orthogonality class* is a class of the form $^{\perp}M$. A full subcategory given by objects orthogonal to some class M of morphism is an *orthogonal subcategory*.

We have $M^{\perp} = {}^{\perp}(M^{\mathrm{op}})$, where $M^{\mathrm{op}} \subseteq \operatorname{Mor} \mathcal{C}^{\mathrm{op}}$ is the class of morphisms in the opposite category determined by M. Consequently, all results about left-orthogonality classes hold for right-orthogonality classes after appropriate dualization.

Proposition 2.1.2. *The operation* $-^{\perp}$ *is left adjoint to* $^{\perp}-$*, in the sense that*

$$M^{\perp} \subseteq N \iff M \supseteq {}^{\perp}N$$

for all classes $M, N \subseteq \operatorname{Mor} \mathcal{C}$ *of morphisms.*

Proof. We have $M \subseteq {}^{\perp}(M^{\perp})$ and $({}^{\perp}N)^{\perp} \supseteq N$ for all classes $M, N \subseteq \operatorname{Mor}\mathcal{C}$, which defines unit and counit of the desired adjunction. ☐

Definition 2.1.3. An *orthogonal system* is a pair (L, R) of classes $L, R \subseteq \operatorname{Mor}\mathcal{C}$ of morphisms such that $L^{\perp} = R$ and $L = {}^{\perp}R$.

Fix an orthogonal system (L, R) for the remainder of this section.

Proposition 2.1.4. *The class L enjoys the following closure properties.*

(i) L contains all isomorphisms.

(ii) If $f \in L$ and $g \in L$, then $gf \in L$ if the composite gf exists.

(iii) If $gf \in L$ with f an epimorphism, then $g \in L$.

(iv) L is stable under pushout. In other words, if

$$\begin{tikzcd} A \arrow[r] \arrow[d, "l"] & A' \arrow[d, "l'"] \\ B \arrow[r] & B' \end{tikzcd}$$

is a pushout square and $l \in L$, then $l' \in L$.

(v) L is stable under colimits. In other words, if I is a small category and $\mu : D \to E$ is a natural transformation of functors $D, E : I \to \mathcal{C}$ such that the components $\mu_i : D(i) \to E(i)$ are in L for all $i \in \operatorname{Ob} I$, then the induced map $\operatorname{colim}\mu : \operatorname{colim} D \to \operatorname{colim} E$ is in L (if the respective colimits exist).

Proof. Isomorphisms are orthogonal to *arbitrary* morphisms, hence *(i)*.

(ii). Let

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "f"] & X \arrow[dd, "r"] \\ B \arrow[d, "g"] & \\ C \arrow[r, "b"] & Y \end{tikzcd} \tag{2.1}$$

be a commutative diagram and suppose that both $f \in L$ and $g \in L$, and that $r \in R$. We obtain a morphism $u : B \to X$ such that $uf = a$ and $ru = bg$ because $f \perp r$, and subsequently $v : C \to X$ such that $vg = u$ and $rv = b$ because $g \perp r$. Then also $vgf = uf = a$, so v commutes with (2.1). If $v' : C \to X$ is another morphism that commutes with (2.1), then $gv' = u$ because $f \perp r$ and thus $v' = v$ because $g \perp r$.

(iii). Let

$$\begin{tikzcd} A \arrow[d, "f"] \arrow[dr, "af"] & \\ B \arrow[r, "a"] \arrow[d, "g"] & X \arrow[d, "r"] \\ C \arrow[r, "b"] & Y \end{tikzcd} \tag{2.2}$$

be a commutative diagram and suppose $gf \in L$, $r \in R$ and that f is epi. We obtain a morphism $u : C \to X$ such that $af = ugf$ and $ru = b$ because $gf \perp r$. Because f is epi, $f = ug$, so u is a filler of the lower square in (2.2). Any two fillers of the lower square are clearly also fillers of the outer pentagon and hence equal.

(iv). Let

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "l"] & A' \arrow[r, "a'"] \arrow[d, "l'"] & X \arrow[d, "r"] \\ B \arrow[r, "b"] & B' \arrow[r, "b'"] & Y \end{tikzcd}$$

be a commutative diagram such that $l \in L$, $r \in R$ and such that the left square is a pushout square. Because $l \perp r$, there is a unique $u : B \to X$ that commutes with the outer rectangle. In particular, $ul = a'a$, so we obtain a unique morphism $u' : B' \to X$ such that $a' = u'l'$ and $u = u'b$. From the latter, we deduce $ru'b = ru = b'b$. It follows that $u'b = b'$ by the universal property of B', for both morphisms induce factorizations of ra' and $b'b$ via b and l'. Clearly every lift $\tilde{u} : B' \to X$ of the right square satisfies $a' = \tilde{u}l'$ and $u = \tilde{u}b$, the latter because any two lifts of the outer diagram agree. Thus, $\tilde{u} = u'$.

(v). Let I, D, E and μ be as in the proposition. Let

$$\begin{tikzcd} \operatorname{colim} D \arrow[r, "a"] \arrow[d, "\operatorname{colim} \mu"'] & X \arrow[d, "r"] \\ \operatorname{colim} E \arrow[r, "b"] & Y \end{tikzcd} \tag{2.3}$$

be a commutative square with $r \in R$. For every object $i \in I_0$ the rectangle

$$\begin{tikzcd} D(i) \arrow[r, "d_i"] \arrow[d, "\mu_i"'] & \operatorname{colim} D \arrow[r, "a"] & X \arrow[d, "r"] \\ E(i) \arrow[r, "e_i"'] \arrow[urr, dashed, "u_i"] & \operatorname{colim} E \arrow[r, "b"'] & Y \end{tikzcd}$$

commutes, and we obtain unique lifts u_i as indicated because $\mu_i \in L$. By the uniqueness of the lifts, the u_i determine a natural transformation $u : E \to \underline{X}$ to the constant functor $\underline{X}$ on D with value X. Consequently, we obtain a unique morphism $u' : \operatorname{colim} E \to X$ such that $u'e_i = u_i$ for all i. Every morphism $\tilde{u} : \operatorname{colim} E \to X$ that commutes with the right-hand square determines lifts $\tilde{e}_i : E(i) \to X$ of the outer rectangle. Thus, $u'e_i = \tilde{e}_i$ and hence $u' = \tilde{u}$. We have therefore no choice but prove that u' commutes with the right rectangle.

b is induced by the universal property of $\operatorname{colim} E$ and the maps $be_i : E(i) \to Y$. But $be_i = ru_i = ru'e_i$, so it follows that $ru' = b$. A similar argument using the universal property of $\operatorname{colim} D$ shows that also $u' \operatorname{colim} \mu = a$, so u' is a diagonal lift in (2.3). ☐

Definition 2.1.5. A *factorization system* is an orthogonal system (L, R) such that furthermore every morphism $f : X \to Y$ in $\mathcal{C}$ can be factored as

$$\begin{tikzcd} X \arrow[r, "l"] \arrow[rr, bend left, "f"] & X' \arrow[r, "r"] & Y \end{tikzcd}$$

with $l \in L$ and $r \in R$.

Factorizations $f = rl$ with $l \in L$ and $r \in R$ are unique up to isomorphism as the following proposition shows. Note that it applies to arbitrary orthogonal systems, but is more powerful for factorization systems, where the factorizations always exist.

Proposition 2.1.6. *For every commutative rectangle of solid arrows*

$$\begin{tikzcd}
X_1 \arrow[r, "l_1"] \arrow[d] \arrow[rr, bend left, "f_1"] & X_1' \arrow[r, "r_1"] \arrow[d, dashed, "\exists! u"] & Y_1 \arrow[d] \\
X_2 \arrow[r, "l_2"] \arrow[rr, bend right, "f_2"'] & X_2' \arrow[r, "r_2"] & Y_2
\end{tikzcd} \tag{2.4}$$

with $l_i \in L$ and $r_i \in R$ there is a unique morphism $u : X_1' \to X_2'$ making the whole diagram commute. □

Proof. Rearrange (2.4) to

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d, "l_1"] & X_2 \arrow[r, "l_2'"] & X_2' \arrow[d, "r_2"] \\
X_1' \arrow[r, "r_1"] & Y_1 \arrow[r] & Y_2.
\end{tikzcd}$$

□

Corollary 2.1.7. *Suppose that (L, R) is an factorization system. Let $Y \in \operatorname{Ob}\mathcal{C}$. Then the full subcategory of the slice category $\mathcal{C}_{/Y}$ spanned by the morphisms in R with codomain Y is reflective.* □

Proof. Factor an arbitrary map $f_1 : X_1 \to Y$ as $f_1 = r_1 l_1$ with $l_1 \in L$ and $r_1 \in R$. Let $f_2 : X_2 \to S$ be morphism in R. Set $r_2 = f_2$ and $l_2 = \mathrm{id}_{X_2}$. By 2.1.6, every map $f_1 \to f_2$ in the slice category $\mathcal{C}_{/Y}$ factors uniquely via f_2. □

Specializing 2.1.7 to $Y = \top$ a terminal object, we have proved that that the full subcategories given by objects right-orthogonal to R is a reflective subcategory, i.e. that the inclusion functor is a right adjoint.

2.2 The orthogonal-reflection construction

Definition 2.2.1. Let M be class of morphisms in $\mathcal{C}$. A commutative square

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[d] & X' \arrow[d] \\
Y \arrow[r] & Y'
\end{tikzcd}$$

in $\mathcal{C}$ is called

- *M-extending* if for every commutative square

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "m"] & X \arrow[d] \\
B \arrow[r] & Y
\end{tikzcd}$$

with $M \in L$ there is a morphism $u : B \to X'$ such that

$$\begin{tikzcd} A \arrow[r] \arrow[d, "m"] & X \arrow[r, "a"] & X' \arrow[d] \\ B \arrow[r] \arrow[urr, "u"] & Y \arrow[r] & Y'; \end{tikzcd}$$

commutes; and

- *M-collapsing* if for every commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "m"] & X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd} \tag{2.5}$$

with $m \in M$ and morphisms $u_1, u_2 : B \rightrightarrows X$ that each (separately) commute with (2.5), we have $au_1 = au_2$.

Definition 2.2.2. An object $X \in \operatorname{Ob}\mathcal{C}$ is called *finitely presentable* if its covariant Hom functor $\operatorname{Hom}(X, -) : \mathcal{C} \to \mathrm{Set}$ preserves filtered colimits.

Let $D : I \to \mathcal{C}$ be functor with I small and filtered and let X be finitely presentable. Spelling out the definition, we find that

- every morphism $f : X \to \operatorname{colim} D$ is represented by a map $X \to D(i)$ for some $i \in \operatorname{Ob} I$; and
- for every commutative square

$$\begin{tikzcd} & D(i) \arrow[dr] & \\ X \arrow[ur, "f"] \arrow[dr, "g"'] & & \operatorname{colim} D \\ & D(j) \arrow[ur] & \end{tikzcd}$$

there exist maps $i \to k$ and $j \to k$ in I for some $k \in \operatorname{Ob} I$ such that

$$\begin{tikzcd} & D(i) \arrow[dr] & \\ X \arrow[ur, "f"] \arrow[dr, "g"'] & & D(k) \\ & D(j) \arrow[ur] & \end{tikzcd}$$

commutes.

Conversely, if X satisfies these conditions for all filtered diagrams D, then X is finitely presentable.

Proposition 2.2.3. *Let M be a class of morphisms with finitely presentable domains and codomains. Let $D, E : I \rightrightarrows \mathcal{C}$ be filtered diagrams in $\mathcal{C}$ and let $\mu : D \Rightarrow E$ be a natural transformation such that for every $i \in \operatorname{Ob} I$ there are morphisms $i \to j$ and $i \to k$ such that the naturality squares*

$$\begin{tikzcd}
D(i) \arrow[r] \arrow[d, "\mu_i"] & D(j) \arrow[d, "\mu_j"] \\
E(i) \arrow[r] & E(j)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
D(i) \arrow[r] \arrow[d, "\mu_i"] & D(k) \arrow[d, "\mu_k"] \\
E(i) \arrow[r] & E(k)
\end{tikzcd}$$

are M-extending and M-collapsing, respectively. Then $\operatorname{colim}\mu : \operatorname{colim} D \to \operatorname{colim} E$ *is in* $M^{\perp}$.

Proof. Let

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "l"] & \operatorname{colim} D \arrow[d, "\operatorname{colim}\mu"] \\
B \arrow[r, "b"] & \operatorname{colim} E
\end{tikzcd} \tag{2.6}$$

be a commutative square with $l \in L$. Because A and B are finitely presentable, a is represented by a map $a : A \to D(i_a)$ and b by a map $b : B \to D(i_B)$. Because I is filtered, there are morphisms morphisms $i_a \to i$ and $i_b \to i$ for some i, so that a and b factor via $D(i)$ and $E(i)$, respectively. Increasing i further, we may also assume that

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "l"] & D(i) \arrow[d, "\mu_i"] \\
B \arrow[r, "b"] & E(i)
\end{tikzcd}$$

commutes. By assumption, there is a morphism $i \to j$ such that the corresponding naturality square for μ is M-extending. Thus, there is a morphism $u : B \to D(j)$ such that

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "l"] & D(i) \arrow[r] & D(j) \arrow[d, "\mu_j"] \\
B \arrow[r, "b"'] \arrow[urr, "u"] & E(i) \arrow[r] & E(j)
\end{tikzcd}$$

commutes. u represents a lift for the commutative square (2.6).

Now suppose that there is another diagonal lift for (2.6). Because A and B are finitely presentable, we may assume that t is represented by the diagonal of the diagram

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "l"] & D(j') \arrow[d, "\mu_{j'}"] \\
B \arrow[r, "b"'] \arrow[ur, "t"] & E(j')
\end{tikzcd}$$

for some j' and that it commutes with the square. Because I is filtered, we find morphisms $j \to \tilde{j}$ and $j' \to \tilde{j}$ for some $\tilde{j} \in \operatorname{Ob} I$, so we may just as well assume that $j = j' = \tilde{j}$. By assumption, there is a map $k \to k'$ in I such that the corresponding naturality square is M-collapsing. It follows that

$$\begin{tikzcd}
 & D(j) \arrow[dr] & \\
B \arrow[ur, "t"] \arrow[dr, "s"] & & D(k) \\
 & D(j) \arrow[ur] &
\end{tikzcd}$$

commutes and that s and t represent the same morphism $B \to \operatorname{colim} D$. ☐

Corollary 2.2.4. *Suppose that there is a class M of morphisms with finitely presentable domains and codomains such that $M^{\perp} = R$. Then R is stable under filtered colimits.*

Proof. Degenerated squares

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "r"] & X \arrow[d, "r"] \\
Y \arrow[r, equal] & Y
\end{tikzcd}$$

with $r \in R$ are trivially both M-extending and M-collapsing. ☐

Lemma 2.2.5. *Suppose that $\mathcal{C}$ is cocomplete. Let $M \subseteq L$ be a set contained in L. Then for every morphism $f : X \to Y$ in $\mathcal{C}$ there exist commutative squares of the form*

$$\begin{tikzcd}
X \arrow[r, "l"] \arrow[d, "f"] & X' \arrow[d, "f'"] \\
Y \arrow[r, equal] & Y
\end{tikzcd} \tag{2.7}$$

with $l \in L$ that are, respectively,

(i) M-extending

(ii) M-collapsing; and

(iii) both M-extending and M-collapsing.

In case (i), l can be constructed as a pushout of a coproduct of morphisms in M.

Proof. *(i)*. Let I be a set of indices $i \in I$ for the set of commutative squares

$$\begin{tikzcd}
A_i \arrow[r, "a_i"] \arrow[d, "m_i"] & X \arrow[d, "f"] \\
B_i \arrow[r, "b_i"] & Y.
\end{tikzcd}$$

with $m_i \in M$. Construct a pushout square

$$\begin{tikzcd}
\coprod_i A_i \arrow[r, "\coprod_i m_i"] \arrow[d, "\langle a_i \rangle_i"'] & \coprod_i B_i \arrow[d] \\
X \arrow[r] & X'
\end{tikzcd}$$

where the top horizontal arrow $\coprod_i m_i$ is given by m_i on the ith component and the left vertical arrow $\langle a_i \rangle_i$ is given by a_i on the ith component.

Denote the lower horizontal morphism by $l : X \to X'$. From *(v)* and *(iv)* of proposition 2.1.4, it follows that $l \in L$. There is a canonical morphism $f' : X' \to Y$ induced by $f : X \to Y$ and the morphism $\langle b_i \rangle_i : \coprod_i B_i \to Y$ given by b_i on the ith copy of B, and it follows by construction that $f = f'l$. By definition of the indexing set I, the square (2.7) is M-extending.

(ii). Let I be a set of indices $i \in I$ for all diagrams

$$\begin{tikzcd}
A_i \arrow[r, "a_i"] \arrow[d, "m_i"'] & X \arrow[d, "f"] \\
B_i \arrow[r, "b_i"'] \arrow[ur, bend left, "u_i"] \arrow[ur, bend right, "v_i"'] & Y.
\end{tikzcd}$$

with $m_i \in M$ that commute after removing either u_i or v_i, i.e. u_i and v_i are (not necessarily equal) lifts for the outer commutative square. Construct a coequalizer diagram

$$\begin{tikzcd} \coprod_i B_i \arrow[r, shift left, "\langle u_i \rangle_i"] \arrow[r, shift right, "\langle v_i \rangle_i"'] & X \arrow[r, "l"] & X' \end{tikzcd}$$

where the morphisms $\langle u_i \rangle_i$ and $\langle v_i \rangle_i$ are given by u_i and v_i, respectively, on the ith component.

Let us prove that $l \in L$. Consider diagonal lifts in the outer rectangle in a commutative diagram

$$\begin{tikzcd} \coprod_i A_i \arrow[r, "\langle a_i \rangle_i"] \arrow[d, "\coprod_i m_i"'] & X \arrow[r, "a"] \arrow[d, "l"] & Z \arrow[d, "g"] \\ \coprod_i B_i \arrow[r, shift left, "l\langle u_i \rangle_i"] \arrow[r, shift right, "l\langle v_i \rangle_i"'] & X' \arrow[r, "b"] & W \end{tikzcd}$$

with $g \in R$. Clearly both $a\langle u_i \rangle_i$ and $a\langle v_i \rangle_i$ are such lifts, so it follows from $\amalg_i m_i \in L$ and orthogonality that $a\langle u_i \rangle_i = a\langle v_i \rangle_i$. Thus, there is a unique factorization of a via l by the universal property of the coequalizer and there is a unique lift for the right square. We conclude $l \in L$. Clearly

$$f\langle u_i \rangle = \langle b_i \rangle = f\langle v_i \rangle,$$

so we obtain a morphism $f' : X' \to Y$ such that $f = f'l$. By definition of I, (2.7) is an M-collapsing factorization.

(iii). Denote by $f : X \xrightarrow{f_1} X_1' \xrightarrow{l_1} Y$ the factorization constructed in *(i)* and let $f : X \xrightarrow{f_2} X_2' \xrightarrow{l_2} Y$ be the factorization constructed in *(ii)*. Consider a pushout square

$$\begin{tikzcd} X \arrow[r, "l_1"] \arrow[d, "l_2"] & \overline{X}_1 \arrow[d] \\ \overline{X}_2 \arrow[r] & X' \end{tikzcd}$$

and let $l : X \to X'$ be the canonical morphism. L is stable under pushout and composition, thus $l \in L$. There is a canonical morphism $f' : X' \to Y$ induced by f_1 and f_2, and clearly (2.7) is M-extending and M-collapsing. □

Definition 2.2.6. A *sequence* in $\mathcal{C}$ is a functor $\mathbb{N} \to \mathcal{C}$, i.e. a diagram

$$\begin{tikzcd} X_0 \arrow[r] & X_1 \arrow[r] & X_2 \arrow[r] & \ldots \end{tikzcd}$$

An *infinite composite* of a sequence $(X_n)_{n\in\mathbb{N}}$ is the canonical map $X_0 \to X_\infty \cong \operatorname{colim}_n X_n$ to a colimit.

Theorem 2.2.7. *Suppose that $\mathcal{C}$ is cocomplete and that there is a set M of morphisms with finitely presentable domains and codomains such that $R = M^\perp$. Then (L, R) is an factorization system. If every morphism in M is an epimorphism, then every morphism in L can be obtained as infinite composition of a sequence of pushouts of coproducts of morphisms in M.*

Proof. Let $f : X \to Y$ be an arbitrary morphism in $\mathcal{C}$. Starting with $X_0 = X$ and $f_0 = f$, we construct inductively commutative squares

$$\begin{tikzcd} X_n \arrow[r, "l_n"] \arrow[d, "f_n"] & X_{n+1} \arrow[d, "f_{n+1}"] \\ Y \arrow[r, equal] & Y \end{tikzcd} \tag{2.8}$$

for all $n \in \mathbb{N}$ that are both M-extending and M-collapsing as in 2.2.5 *(iii)*.

By 2.1.4 *(v)*, L is stable under colimits. It follows that the infinite composition

$$l \coloneqq \operatorname{colim}_n \, (l_{n-1} \circ \dots \circ l_0) : \operatorname{colim}_n X \xrightarrow{\sim} X \to \operatorname{colim}_n X_n$$

is in L. Clearly the total order $\mathbb{N}$ is, as a category, filtered. But then 2.2.3 is applicable and

$$r \coloneqq \operatorname{colim} f_n : \operatorname{colim}_n X_n \to \operatorname{colim}_n Y \xrightarrow{\sim} Y$$

is in R. Thus, we have factored $f = rl$ with $r \in R$ and $l \in L$.

If M contains epimorphisms only, we may construct the commutative squares (2.8) as in 2.2.5 *(i)* because *every* commutative square is automatically M-collapsing. Now if f is already in L, then $f = \mathrm{id} \circ f$ is a factorization with $\mathrm{id} \in R$ and $f \in L$. Because such factorizations are unique up to isomorphism by 2.1.6, r is an isomorphisms. Thus, $f = rl$ is the infinite composition of the sequence $X_0 \to X_1 \to \dots$ of morphisms which have been obtained via 2.2.5 *(i)*, i.e. as pushouts of coproducts of morphisms in M. $\square$

Corollary 2.2.8. *In the situation of 2.2.7, suppose that M consists of epimorphisms only. Let N be a class of morphisms in $\mathcal{C}$ that contains all pushouts of coproducts of morphisms in M and is closed under infinite composition. Then $L \subseteq N$.* $\square$

3 Partial algebras and partial Horn logic

This section is devoted to the study of categories of *partial* Σ*-algebras* over multi-sorted signatures Σ [2] and certain full subcategories thereof, the categories of models of *partial Horn logic* [17] theories.

3.1 Categories of partial algebras

We begin with the analysis of basic category theoretic properties of partial algebras, their (co)limits and free partial algebras arising from signature morphisms; most results can also be found in [2] or follow easily from there.

Definition 3.1.1. Let S be a small set. An S*-sorted set* is a morphism $f : X \to S$. A morphism $h : f \to g$ of S-sorted sets is a morphism in the slice category $\mathrm{Set}_{/S}$.

We will usually write $(X_s)_{s\in S}$ for the S-sorted set $\coprod_{s\in S} X_s \to S$. Then a morphism $f : (X_s)_{s\in S} \to (Y_s)_{s\in S}$ of S-sorted sets is a family of functions $f = (f_s : X_s \to Y_s)_{s\in S}$.

Definition 3.1.2. Let A and B be sets. A *partial function from* A *to* B is a function $f : U \to B$ such that $U \subseteq A$. In this situation, we write $f : A \rightharpoondown B$ and refer to $U = \operatorname{dom} f$ as the *domain* of f. We write $f(x)\downarrow$ and say f is *defined* on an element $x \in A$ if $x \in \operatorname{dom} f$.

Definition 3.1.3. A *signature* is a datum $\Sigma = (S, P, \mathrm{ar})$, where

- S and P are sets, which we think of as sets of *sort symbols* and *operation symbols*; and
- $\mathrm{ar} : P \to \coprod_{n\geq 0} S^{n+1}$ assigns each operation $p \in \Sigma$ a non-empty finite list $\mathrm{ar}\, p = (s_1, \ldots, s_n, s)$ of elements of S, the *arity* of the operation p.

For operation symbols p and sorts $s_1, \ldots, s_n, s$, we write

$$p : s_1 \times \cdots \times s_n \to s$$

if $\mathrm{ar}\, p = (s_1, \ldots, s_n, s)$. If $n = 0$, we omit the arrow and instead write $p : s$.

Fix a signature Σ for the remainder of this section.

Definition 3.1.4. We define a category $\mathrm{Palg}(\Sigma)$ of *partial* Σ*-algebras* and their morphisms as follows. A partial Σ-algebra X consists of an S-sorted set $(X_s)_{s\in S}$ and partial functions $p_X : X_{s_1} \times \cdots \times X_{s_n} \rightharpoondown X_s$ for each operation symbol $p : s_1 \times \cdots \times s_n \to s$. A *morphism* $f : X \to Y$ of partial Σ-algebras is a family of (total) functions $f_s : X_s \to Y_s$ for each $s \in S$ that commute with the partial functions induced by the operation symbols, in the sense that

$$f_s(p_X(x_1, \ldots, x_n)) = p_Y(f_{s_1}(x_1), \ldots, f_{s_n}(x_n)) \tag{3.1}$$

for all operation symbols $p : s_1 \times \cdots \times s_n \to s$ and $(x_1, \ldots, x_n) \in \operatorname{dom} X_p \subseteq X_{s_1} \times \cdots \times X_{s_n}$. In particular,

$$p_X(x_1, \ldots, x_n)\downarrow \implies p_Y(f_{s_1}(x_1), \ldots, f_{s_n}(x_n))\downarrow .$$

In order to improve the readability of formulae, we will often use more concise but slightly ambiguous notation if confusion is unlikely: We write $f(x)$ instead of $f_s(x)$ if $x \in X_s$. Likewise, if the sorts of elements $x_1 \in X(s_1), \ldots, x_n \in X(s_n)$ are clear, we omit them from the notation and instead write $x_1, \ldots, x_n \in X$. In cases where there is no doubt about the partial algebra in question, we write $p(x_1,\dots,x_n)$ instead of $p_X(x_1,\dots,x_n)$. For example, (3.1) can be more succinctly written as $f(p(x_1,\dots,x_n)) = p(f(x_1),\dots,f(x_n))$ for all $x_1,\dots,x_n \in X$.

The discussion of limits and (some) colimits in $\mathrm{Palg}(\Sigma)$ is perhaps most elegantly done by identifying partial Σ-algebras with certain functors $\mathcal{C}_\Sigma \to \mathrm{Set}$, where $\mathcal{C}_\Sigma$ is the category defined as follows. Each sort $s \in S$ is an object of $\mathcal{C}_\Sigma$, and for each operation symbol $p : s_1 \times \cdots \times s_n \to s$, there are objects $(s_1,\dots,s_n)$ and $\mathrm{dom}\, p$ in $\mathcal{C}_\Sigma$. (here, $\mathrm{dom}\, p$ are arbitrary but distinctly chosen symbols.) Apart from identity morphisms, there are morphisms $i_p : \mathrm{dom}\, p \to (s_1,\dots,s_n)$ and $p : \mathrm{dom}\, p \to s$ for each operation symbol $p : s_1 \times \cdots \times s_n \to s$. The reader can now verify that $\mathcal{C}_\Sigma$ is indeed a category (note that we do not identify a length-one tuple (s) with its sole element s), although in what follows, $\mathcal{C}_\Sigma$ can also be treated as a graph.

Partial algebras can now be described as functors $X : \mathcal{C}_\Sigma \to \mathrm{Set}$ such that $X((s_1,\dots,s_n)) = X(s_1)\times\cdots\times X(s_n)$ and $X(i_p) : X(\mathrm{dom}\, p) \hookrightarrow X((s_1,\dots,s_n))$ is an injection for all operation symbols $p : s_1\times\cdots\times s_n \to s$. Morphisms of partial Σ-algebras correspond to natural transformations of such functors.

Proposition 3.1.5. $\mathrm{Palg}(\Sigma)$ *is complete. The carrier functor preserves limits. A morphism f in* $\mathrm{Palg}(\Sigma)$ *is a monomorphism if and only if* $\mathrm{car}\, f$ *is a monomorphism.*

Proof. Because limits preserve products and monomorphisms, the limit of a diagram of partial Σ-algebras as computed in $\mathrm{Set}^{\mathcal{C}_\Sigma}$ is again a partial Σ-algebra and hence a limit in $\mathrm{Palg}(\Sigma)$.

The carrier functor is a right adjoint and hence preserves limits. Morphisms of partial Σ-algebras are maps of underlying carrier sets, so if $\mathrm{car}\, f$ is a monomorphism then so is f. For the converse, note that $\mathrm{Palg}(\Sigma)$ and $\mathrm{Set}_{/S}$ are complete and car is a right adjoint and hence continuous. Because monomorphisms can be detected by whether certain squares are pullback squares, it follows that car preserves monomorphisms. ☐

Definition 3.1.6. Let X be a partial Σ-algebra. A *congruence* $R = (R_s)_{s\in S}$ on X is a family of equivalence relations $R_s \subseteq X_s \times X_s$ such that for each operation $p : s_1 \times \cdots \times s_n \to s$ and elements $(x_1, y_1) \in R_{s_1}, \dots, (x_n, y_n) \in R_{s_n}$ satisfying $p(x_1,\dots,x_n)\downarrow$ and $p(y_1,\dots,y_n)\downarrow$, it holds that

$$(p(x_1,\dots,x_n), p(y_1,\dots,y_n)) \in R(s).$$

Lemma 3.1.7. *Let $R_0 = (R_0)_{s\in S}$ be a family of relations $(R_0)_s \subseteq X_s \times X_s$ for some partial Σ-algebra X. Then there is a least congruence R on X that contains R_0.*

Proof. This follows from the fact that arbitrary componentwise intersections of congruences are congruences, and that the family of maximal relations on each X_s is a congruence. ☐

Lemma 3.1.8. *Let X be partial Σ-algebra, let R_0 be a relation on* $\mathrm{car}\, X$ *and let R be the least congruence containing R_0. Then the partial Σ-algebra $X/R_0 = X/R$ given by $(X/R)_s = X_s/R_s$ and the partial functions*

$$p_{X/R}([x_1],\dots,[x_n]) = [p_X(x_1,\dots,x_n)] \tag{3.2}$$

for all operation symbols p and $x_1,\dots,x_n \in \mathrm{dom}\, p_X$ is well-defined. The canonical morphism $f : X \to X/R$ has the following universal property:

Every morphism $g : X \to Y$ satisfying $f(x) = f(y)$ for all $(x, y) \in R_0$ can be factored uniquely via f, i.e there exists a unique morphism $h : X/R \to Y$ such that

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "f"'] & Y \\
X/R \arrow[ur, dashed, "\exists! h"'] &
\end{tikzcd} \tag{3.3}$$

commutes.

Proof. The well-definedness of Y/R is immediate from the definition of a congruence. For every morphism g as in (3.3), we can factor $\operatorname{car} g$ uniquely via $\operatorname{car} f$ because the family of kernel relations $\{(x, y) \in X_s \times X_s \mid f(x) = f(y)\}$ is a congruence and hence contains R. Thus, $\operatorname{car} h$ is uniquely determined. Given (3.2), it is clear that $\operatorname{car} h$ is compatible with the operations and hence arises from a morphism h of partial Σ-algebras. ☐

Proposition 3.1.9. $\operatorname{Palg}(\Sigma)$ *is cocomplete and closed under filtered colimits in* $\operatorname{Set}^{\mathcal{C}_\Sigma}$*.*

Proof. Filtered colimits preserve products and monomorphism in Set (in fact all finite limits). and hence also in $\operatorname{Set}^{\mathcal{C}_\Sigma}$. It follows that $\operatorname{Palg}(\Sigma)$ is closed in $\operatorname{Set}^{\mathcal{C}_\Sigma}$ under filtered colimits.

Let $(X_i)_{i \in I}$ be a family of partial Σ-algebras. We will construct its coproduct $Y = \coprod_i X_i$ as follows. For each $s \in S$, we set

$$Y_s = \coprod_i (X_i)_s.$$

The operations are given by

$$p_Y(x_1, \dots, x_n) = p_{X_i}(x_1, \dots, x_n) \in X_i \hookrightarrow \operatorname{car} Y \tag{3.4}$$

for all $x_1, \dots, x_n \in X_i$ for some $i \in I$ such that the left-hand side of (3.4) is defined, and undefined otherwise. Thus, if $x_1, \dots, x_n \in \coprod_i X_i$ are not elements of the same component, $p(x_1, \dots, x_n)$ is undefined. The maps $\operatorname{car} X_i \to \operatorname{car} Y$ are clearly maps of partial Σ-algebras. Likewise, the map of S-sorted sets $(Y_s)_s \to (Z_s)_s$ induced by morphisms of partial Σ-algebras $X_i \to Z$ to some partial Σ-algebra Z is compatible with the operations. It follows that Y is a coproduct of $(X_i)_{i \in I}$ in $\operatorname{Palg}(\Sigma)$.

It remains the construction of coequalizer of pairs $f, g : X \rightrightarrows Y$. This is easily done using congruences: Let $R_0 = \{(f(x), g(x)) \mid x \in X\}$. Then the quotient Y/R_0 together with the canonical projection $Y \to Y/R_0$ as constructed in lemma 3.1.8 is a coequalizer of f and g. ☐

Corollary 3.1.10. *A morphism $f : X \to Y$ is an effective epimorphism if and only if* $\operatorname{car} f$ *is surjective and the induced map* $\operatorname{dom} p_X \to \operatorname{dom} p_Y$ *given by restriction of f is surjective for all operation symbols $p \in P$.*

Proof. f is an effective epimorphism if and only if it is isomorphic to the canonical map $X \to X/R$ for some congruence R on X. ☐

Corollary 3.1.11. $\operatorname{Palg}(\Sigma)$ *is a regular category, i.e. regular epimorphisms are stable under pullback.*

Proof. Every regular epimorphism is isomorphic to a map $X \to X/R$ for a congruence R on X, which is a surjection of the carrier sets by definition.

The stability of effective epimorphisms under pullbacks follows from the characterization 3.1.10 and the stability of surjections under pullbacks in Set. For, pullbacks in $\operatorname{Palg}(\Sigma)$ are computed as pullbacks in $\operatorname{Set}^{\mathcal{C}_\Sigma}$ and hence pointwise in Set. ☐

Definition 3.1.12. Let $\Sigma = (S, P, \operatorname{ar})$ and $\Sigma' = (S', P', \operatorname{ar}')$ be signatures. A *signature morphism* $F : \Sigma \to \Sigma'$ consists of maps $S \to S'$ and $P \to P'$ (which we both denote by F) that are compatible with ar and ar′, in the sense that

$$p : s_1 \times \dots \times s_n \implies F(p) : F(s_1) \times \dots F(s_n) \to F(s)$$

for all operation symbols p and sort symbols $s_1, \dots, s_n, s$ in Σ.

Trivial examples for signature morphisms are component-wise inclusions $\Sigma \subseteq \Sigma'$ which are compatible with the arity functions. In this case, we also say that Σ' is an *extension* of Σ.

Every signature morphism $F : \Sigma \to \Sigma'$ induces a functor $\mathcal{C}_\Sigma \to \mathcal{C}_{\Sigma'}$ and consequently, via precomposition, a functor $\mathrm{Set}^{\mathcal{C}_{\Sigma'}} \to \mathrm{Set}^{\mathcal{C}_\Sigma}$, which restricts to a functor $F_* : \mathrm{Palg}(\Sigma') \to \mathrm{Palg}(\Sigma)$. Explicitly, if $X' \in \mathrm{Palg}(\Sigma')$, then

$$F_*(X')_s = X'_{F(s)},$$

for each $s \in S$ and

$$p_{F_*(X')}(x_1, \ldots, x_n) = F(p)_{X'}(x_1, \ldots, x_n).$$

for all operation symbols $p : s_1 \times \cdots \times s_n \to s$ in Σ.

Definition 3.1.13. Let $\Sigma = (S, P, \mathrm{ar})$ be a signature. The functor

$$\mathrm{car} := F_* : \mathrm{Palg}(\Sigma) \to \mathrm{Set}_{/S}$$

induced by the signature morphism $F : (S, \emptyset, \emptyset) \to \Sigma$ is called the *carrier* functor.

Proposition 3.1.14. *Let $F : \Sigma \to \Sigma'$ be a signature morphism. Then there is a functor $F^* : \mathrm{Palg}(\Sigma) \to \mathrm{Palg}(\Sigma')$ and an adjunction $F^* \dashv F_*$. If F is injective on sort symbols and operation symbols, then F^* is full and faithful.*

Proof. We consider the separate cases

(i) F is the identity on operation symbols; and

(ii) F is the identity on sort symbols.

This suffices to prove the proposition because a general signature morphism F can be decomposed as $F = F_2 F_1$ where F_1 satisfies the condition *(i)* and F_2 satisfies *(ii)*. In both cases, we content ourselves with constructing $X' = F^*(X)$ and the unit $\eta : X \to F_*(X')$ for $X \in \mathrm{Ob}\,\mathrm{Palg}(\Sigma)$; the verification of the universal property is easy and hence omitted.

(i). For each sort $s' \in S'$, set

$$X'_{s'} = \coprod_{s \in F^{-1}(s')} X_s.$$

Corresponding to the inclusions onto the respective components of coproducts, we obtain maps

$$\eta_s : X_s \hookrightarrow F_*(X')_s = \coprod_{\substack{\bar{s} \in S \\ F(\bar{s}) = F(s)}} X_{\bar{s}}$$

for each $s \in S$. Let $p : s_1 \times \cdots \times s_n \to s$ be an operation in Σ. Then $\mathrm{dom}\, p_{X'}$ is given by the inclusion

$$\mathrm{dom}\, p_{X'} = p_X \hookrightarrow X_s \overset{\eta_s}{\hookrightarrow} F_*(X')_s = X'_{F(s)}$$

while $p_{X'}$ itself is the function

$$p_{X'} : \mathrm{dom}\, p_X \xrightarrow{p_X} X_{s_1} \times \cdots \times X_{s_n} \xrightarrow{\eta_{s_1} \times \cdots \times \eta_{s_n}} X'_{s_1} \times \cdots \times X'_{s_n}.$$

If F is an inclusion on sort symbols, then F^* is the inclusion of the full subcategory given by the partial Σ'-algebras X' with $X_s = \emptyset$ for all s not in the image of F.

(ii). Let R be the least congruence (see lemma 3.1.7) on X such that

$$(p_X(x_1, \ldots, x_n), q_X(x_1, \ldots, x_n)) \in R$$

for all operations $p, q \in P$ such that $F(p) = F(q)$ and $x_1, \ldots, x_n$ in the domains of both p_X and q_X. Note that by the assumption on F, from $F(p) = F(q)$ it follows that $\mathrm{ar}(p) = \mathrm{ar}(q)$. Now $X' = F_*(X)$ is given by

$$X'_s = X/R_s$$

for all $s \in S$ and

$$p'_{X'}([x_1], \ldots, [x_n]) = [p_X(x_1, \ldots, x_n)]$$

for all operation symbols $p' \in P'$ and all operations p such that $F(p) = p'$ and $p_X(x_1, \ldots, x_n)\downarrow$. Thus, $F_*(X') = X/R$ and η is the canonical map to the quotient. If F is an inclusion on operation symbols, then R is the diagonal congruence. Thus, F^* is the inclusion of the full subcategory given by the partial Σ-algebras X' for which $p_{X'} = \emptyset$ is entirely undefined. ☐

It follows that the carrier functor 3.1.13 is full and faithful; it is the inclusion of the full subcategory of partial Σ-algebras with entirely undefined operations $p_X = \emptyset$. We will thus identify such partial Σ-algebras with their underlying S-sorted sets.

Definition 3.1.15. A partial Σ-algebra X is *finite* if $\coprod_{s\in S} X_s$ and $\coprod_{p\in P} \mathrm{dom}\, p_X$ are finite sets.

Lemma 3.1.16. *Let $\mathcal{C}$ be a category and let $F : \mathcal{C} \to \mathrm{Set}$ be a functor such that $\coprod_{c\in \mathrm{Ob}\,\mathcal{C}} F(c)$ is a finite set. Then F is finitely presentable in $\mathrm{Set}^{\mathcal{C}}$.*

Proof. Recall that a functor $F : \mathcal{C} \to \mathrm{Set}$ can equivalently be described as a map

$$\pi_F : \int F = \coprod_{c\in \mathrm{Ob}\,\mathcal{C}} F(c) \to \mathrm{Ob}\,\mathcal{C}$$

which assigns to each element its component, and a partial function

$$m_F : \mathrm{Mor}\,\mathcal{C} \times \int F \rightharpoonup \int F$$

corresponding to the action of F on morphisms in $\mathcal{C}$. The laws these operations have to satisfy can be found in e.g. [15].

A natural transformation $\mu : F \Rightarrow G$ is then equivalent to a map $\int \mu : \int F \to \int G$ which is compatible with the projections π_F, π_G and the action maps m_F, m_G.

Now let $D : I \to \mathrm{Set}^{\mathcal{C}}$ be a filtered diagram and let $\mu : F \Rightarrow \mathrm{colim}\, D$ be a natural transformation. Colimits in functor categories are computed componentwise, thus

$$\int \mathrm{colim}\, D \cong \mathrm{colim}_i \int D(i).$$

By assumption, $\int F$ is a finite set and hence finitely presentable in Set. It follows that $\int \mu : \int F \to \mathrm{colim}_i \int D(i)$ arises from some map $(\int \mu)_0 : \int F \to \int D(i_0)$ for some $i_0 \in \mathrm{Ob}\, I$. $(\int \mu)_0$ is compatible with the projection maps π_F and $\pi_{D(i_0)}$ but not necessarily with the action maps m_F and $m_{D(i_0)}$.

However, we may assume without loss of generality that the composite map

$$\mathrm{Im}\,(\int \mu)_0 \hookrightarrow \int D(i_0) \to \int \mathrm{colim}\, D \tag{3.5}$$

is injective, increasing i_0 for every two elements $x, y \in \mathrm{Im}(\int \mu)_0$ which are identified by the map (3.5). It is then clear that $(\int \mu)_0$ is compatible with the action maps; all equations that have to hold in $\mathrm{Im}(\int \mu)_i$ hold after application of (3.5) and are by injectivity already true in the preimage. It follows that $(\int \mu)_0$ arises from a natural transformation $\mu_0 : F \Rightarrow D(i_0)$.

Now suppose that μ also arises from some other natural tranformation $\mu_1 : F \Rightarrow D(i_1)$ for some $i_1 \in \mathrm{Ob}\, I$. We find an index i_2 along with morphisms $k_0 : i_0 \to i_2$ and $k_1 : i_1 \to i_2$ such that

$$\begin{tikzcd}
\int F \arrow[r, "\int \mu_0"] \arrow[d, "\int \mu_1"] & \int D(i_0) \arrow[d, "\int D(k_0)"] \\
\int D(i_1) \arrow[r, "\int D(k_1)"] & \int D(i_2)
\end{tikzcd}$$

commutes because $\int F$ is finitely presentable. It follows that $D(k_0) \circ \mu_0 = D(k_1) \circ \mu_1$. ☐

Proposition 3.1.17. *Every finite partial Σ-algebra is finitely presentable.*

Proof. By 3.1.9, filtered colimits in $\mathrm{Palg}(\Sigma)$ can be computed in the functor category $\mathrm{Set}^{\mathcal{C}_\Sigma}$. If X is a partial Σ-algebra, then $\coprod_{c \in \mathcal{C}_\Sigma} X(c)$ is finite if and only if X is finite, thus 3.1.16 applies. ☐

Proposition 3.1.18. *The full subcategory of* $\mathrm{Palg}(\Sigma)$ *spanned by the finite partial Σ-algebras is closed under finite colimits and non-empty finite limits.*

Proof. We can immediately reduce to proving that finite algebras are closed under binary (co)products and (co)equalizers and that the initial partial Σ-algebra is finite. The hardest step is the construction of finite coequalizers; we will content ourselves with proving only this step here. (Notice that, constructively, a quotient of a finite set need not necessarily be decidable and hence not finite!)

Recall from the proof of 3.1.9 that the coequalizer of a parallel pair of morphisms $f, g : X \rightrightarrows Y$ is given by the the canonical map $Y \to Y/R_0$ with $\mathrm{car}\,(Y/R_0) = (\mathrm{car}\, Y)/R$, where

$$(R_0)_s = \{(y_1, y_2) \in Y_s \times Y_s \mid \exists x_1, x_2 \in X_s(f(x_1) = f(x_2))\}.$$

and R is the least congruence on Y containing R_0. Clearly if we can show that the congruence R is finite (as a finite partial Σ-algebra with trivial operations), then Y/R is finite. Because X and Y are finite, we know that R_0 is finite and in particular decidable.

Construct a sequence of relations

$$R_0 \hookrightarrow R_1 \hookrightarrow \dots$$

by defining R_{n+1} as the union of R_n with the set of all tuples

$$(p(x_1, \dots, x_n), p(y_1, \dots, y_n))$$

for operation symbols p such that p_Y is nontrivial and $(x_1, \dots, x_n), (y_1, \dots, y_n) \in \mathrm{dom}\, p_Y$ such that $(x_1, y_1), \dots, (x_n, y_n) \in R_n$.

By induction, it follows that R_n is finite and contained in the congruence R. Clearly $R_n = R_{n+1}$ if R_n is a congruence and $\#R_{n+1} > \#R_n$ otherwise. The size of a relation on $\mathrm{car}\, Y$ is bounded by $n = \sum_{s \in S} (\#Y_s)^2$. Thus, $R_n = R_{n+1} = R$. ☐

3.2 Total, totalizing and saturated morphisms

Partial algebras are more general than *total* algebras, as in the former, the operation symbols given by the signature may be interpreted as partial functions. We analyze the relation between partial and total Σ-algebras. Using the machinery of section 2, we construct the factorization system of *totalizing* and *total* morphisms of partial algebras. Totalizing morphisms can be thought of as "relative" algebras over the codomain, while totalizing morphisms are given by freely making operations of a partial algebra more defined.

Definition 3.2.1. A morphism $f : X \to Y$ of partial Σ-algebras is *total* if for all operation symbols $p : s_1 \times \cdots \times s_n \to s$ and elements $x_1, \ldots, x_n \in X$ it holds that

$$p_Y(f(x_1), \ldots, f(x_n)) \downarrow \Longrightarrow p_X(x_1, \ldots, x_n) \downarrow .$$

A partial Σ-algebra Z is a *total* Σ*-algebra*, or simply Σ*-algebra*, if the unique morphism $! : Z \to \top$ to the terminal partial Σ-algebra $\top$ is total.

Thus, an algebra is a partial Σ-algebra X for which the partial functions p_X are total for all p.

Proposition 3.2.2. *Let $D : I \to \mathrm{Palg}(\Sigma)$ be a diagram such that $D(k)$ is total for each morphism $k \in \mathrm{Mor}\, I$. Then the comparison map $\mathrm{colim}\,(\mathrm{car}\, D) \to \mathrm{car}\,(\mathrm{colim}\, D)$ is an isomorphism.*

Proof. Recall the construction of colimits via coproducts and coequalizers: $\mathrm{colim}\, D$ fits into a coequalizer diagram

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r] & \mathrm{colim}\, D, \end{tikzcd}$$

where

$$X = \coprod_{i \in \mathrm{Ob}\, I} D(i)$$

and

$$Y = \coprod_{\substack{i,j \in \mathrm{Ob}\, I \\ k,\ell : i \to j}} D(i).$$

The map f is given by the maps $k : i \to j$ and g by the maps indexed by ℓ.

From the construction of coproducts in the proof of 3.1.9, it is clear that car commutes with coproducts. It follows that $\mathrm{colim}\,(\mathrm{car}\, D)$ can be constructed as coequalizer

$$\begin{tikzcd} \mathrm{car}\, X \arrow[r, shift left, "\mathrm{car}\, f"] \arrow[r, shift right, "\mathrm{car}\, g"'] & \mathrm{car}\, Y \arrow[r] & \mathrm{colim}\,(\mathrm{car}\, D). \end{tikzcd}$$

From $D(k), D(\ell)$ total for all k, ℓ, it follows that f and g are total. Thus, we have reduced to the case where D *is* is the diagram given by a pair $f, g : X \rightrightarrows Y$.

Again from the proof of 3.1.9, we know that the coequalizer of f and g can be constructed as quotient Y/R, where R is the congruence on Y generated by the relation

$$R_0 = \{(f(x), g(x)) \mid x \in X\}.$$

The coequalizer of $\mathrm{car}\, f$ and $\mathrm{car}\, g$ is the quotient $(\mathrm{car}\, Y)/R_0$. Thus, we have to show that $R = R_0$, i.e. that R_0 is already a congruence.

Let $p : s_1 \times \cdots \times s_n \to s$ be an operation symbol and let $x_1, \ldots, x_n \in X$ such that $p_Y(f(x_1), \ldots, f(x_n)) = y$ and $p_Y(g(x_1), \ldots, g(x_n)) = z$. Because f is a relative algebras, we have $p_X(x_1, \ldots, x_n) = x$ for some x, and because f and g are morphisms of partial Σ-algebras, $f(x) = y$ and $g(x) = z$, hence $(y, z) \in R_0$. ☐

Corollary 3.2.3. *Let $f : X \to Y$ be a total epimorphism. Then f is an effective epimorphism.*

Proof. By 3.1.10, it suffices to verify that f_s is surjective for all s and induces surjections $\operatorname{dom} p_X \to \operatorname{dom} p_Y$ for all p.

f is an epimorphism if and only if

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"] & Y \arrow[d, equal] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$

is a pushout square. By 3.2.2, it follows that car preserves this colimit. But then $\operatorname{car} f$ is an epimorphism in $\mathrm{Set}_{/S}$, i.e. surjective.

Now let $(y_1, \ldots, y_n) \in \operatorname{dom} p_Y$. Because $\operatorname{car} f$ is surjective, there are elements $x_1, \ldots, x_n \in X$ such that $f(x_i) = y_i$ for all i, and because f is total, we have $(x_1, \ldots, x_n) \in \operatorname{dom} p_X$. ☐

For each operation symbol q, let A^q be the partial Σ-algebra given by the S-sorted set

$$A^q_t = \{i \in \{1, \ldots, n\} \mid s_i = t\}$$

and let B^q be the partial Σ-algebra given by

$$B^q_t = \{i \in \{1, \ldots, n+1\} \mid s_i = t\}$$

and partial functions B^q_p non-trivial only for $p = q$, where

$$B^q_q(1, \ldots, n) = n + 1.$$

and undefined for other elements of B^q.

Proposition 3.2.4. *A morphism of partial Σ-algebras is total if and only if it is right-orthogonal to m^q for all operation symbols q.* ☐

Definition 3.2.5. A morphism of partial Σ-algebras is *totalizing* if it is left-orthogonal to all total morphisms.

Clearly A^q and B^q are finite partial Σ-algebras for all q, an in particular finitely presentable. Thus, the classes of totalizing and total morphisms are parts of a factorization system, and the full subcategory of $\mathrm{Palg}(\Sigma)$ spanned by the total Σ-algebras is reflective. We construct pushouts of coproducts of morphisms m^q explicitly. Totalizing morphisms of this form will be referred to as *single-step totalizing.*

Lemma 3.2.6. *Let X be a partial Σ-algebra, let $(q_i)_{i \in I}$ be a family of operation symbols and let $(a_i)_{i \in I}$ be a family of morphisms $a_i : A_{q_i} \to X$. For each sort symbol s, let*

$$I_s = \{i \in I \mid q_i : s_1 \times \cdots \times s_n \to s \text{ for some } s_1, \ldots, s_n \}.$$

Let Y be the partial Σ-algebra given by

$$Y_s = (X_s \amalg I_s)/R_s,$$

where R_s is the least symmetric and reflexive relation such that

$$((q_i)_X(a_i(1), \ldots, a_i(n)), i) \in R_s$$

for all $i \in I_s$ and appropriate n. The operations p_Y are given by the clauses

$$p_Y(x_1, \ldots, x_n) = P_X(x_1, \ldots, x_n)$$

for all p *and* $x_1, \dots, x_n$ *in* X *and*

$$(q_i)_Y(a_i(1), \dots, a_i(n)) = i$$

for all i. *Then* R_s *is an equivalence relation for all* s, Y *is a well-defined partial* Σ*-algebra and*

$$\begin{tikzcd}
\coprod_i A_{q_i} \arrow[r, "\coprod_i m_{q_i}"] \arrow[d, "\langle a_i \rangle_i"] & \coprod_i B_{q_i} \arrow[d] \\
X \arrow[r] & Y
\end{tikzcd}$$

is a pushout square. □

Proposition 3.2.7. *Every totalizing morphism is a monomorphism.*

Proof. Clearly the single-step totalizing morphism $X \to Y$ as constructed in 3.2.6 is a monomorphism. Thus, every pushout of a coproduct of morphisms m^q is a monomorphism. Monomorphisms are stable under filtered coproducts and in particular infinite composition. We conclude with 2.2.8. □

Proposition 3.2.8. *Every totalizing morphism is an epimorphism.*

Proof. All of coproduct, pushout and filtered colimit preserve epimorphisms. Thus, 2.2.8 applies. □

Definition 3.2.9. A morphism $f : X \to Y$ of partial algebra is *saturated* if for all operation symbols $p : s_1 \times \dots \times s_n \to s$ and all elements $x \in X$ and $y_1, \dots, y_n \in Y$ with $p_Y(y_1, \dots, y_n) = f(x)$, there is exactly one tuple $(x_1, \dots, x_n) \in \operatorname{dom} p_X$ such that $f(x_i) = y_i$ for all i and $p_X(x_1, \dots, x_n) = x$.

As was the case for total morphisms, the class of saturated morphisms is a right-orthogonality class. Let $q : s_1 \times \dots \times s_n \to s_{n+1}$ be an operation symbol. Let $(A')^q$ be the partial Σ-algebra given by the single element $n + 1 \in A'_{s_{n+1}}$. There is a canonical inclusion $n^q : (A')^q \to B^q$, where B^q is the codomain of the totalizing morphism m^q defined earlier. The following is then a triviality.

Proposition 3.2.10. *A morphism* f *is saturated if and only if* $n^q \perp f$. *The class of saturated morphisms is a right-orthogonality class.* □

Proposition 3.2.11. *Every totalizing morphism is saturated.*

Proof. We verify the conditions of 2.2.8. The map $X \to Y$ from 3.2.6 is saturated, i.e. pushouts of coproducts of morphisms m^q are saturated. By 2.2.4, the right-orthogonality class of saturated morphisms is closed under filtered colimits. □

Totalizing morphisms are not closed under all pullbacks. For example, for all operation symbols q, the pullback of m^q along the map $\operatorname{car} B^q \to B^q$ is the inclusion $A^q \to \operatorname{car} B^q$ which is not an epimorphism and hence not totalizing. However, pullback along *saturated* morphisms preserves totalizing morphisms.

Proposition 3.2.12. *Let*

$$\begin{tikzcd}
Y' \times_Y X \arrow[r] \arrow[d, "t'"] & X \arrow[d, "t"] \\
Y' \arrow[r, "f"] & Y
\end{tikzcd}$$

be a pullback square in $\mathrm{Palg}(\Sigma)$ *such that* t *is totalizing and* f *is saturated. Then* t' *is totalizing.*

Proof. Pullback and filtered colimits are computed in $\mathrm{Set}^{\mathcal{C}_\Sigma}$ and hence commute. Saturated morphisms are defined by right-orthogonality and are hence stable under pullback. Thus, we reduce to the case of t single-step totalizing and assume that t is given as in 3.2.6. Then

$$(Y' \times_Y X)_s = f^{-1}(X_s) \subseteq Y'_s$$

for each sort $s \in S$ and

$$p_{Y' \times_Y X}(y'_1, \dots, y'_n) = y \iff p_{Y'}(y'_1, \dots, y'_n) = y'$$

for all $y'_1, \dots, y'_n, y' \in Y' \times_Y X$ and operation symbols $p : s_1 \times \cdots \times s_n \to s$. Note that if $p_{Y'}(y'_1, \dots, y'_n) = y'$, then $p_Y(f(y'_1), \dots, f(y'_n)) = f(y')$ and hence $p_X(f(y'_1), \dots, f(y'_n)) = f(y')$ by definition of Y.

Now let $I' = f^{-1}(I)$. We will construct a cocartesian square

$$\begin{tikzcd}
\coprod_{i'} A^{q_{i'}} \arrow[r, "\coprod_{i'} m^{q_{i'}}"] \arrow[d, "\langle a_{i'} \rangle_{i'}"'] & \coprod_{i'} B^{q_{i'}} \arrow[d, "\langle b_{i'} \rangle_{i'}"] \\
Y' \times_Y X \arrow[r, "t'"] & Y',
\end{tikzcd}$$

with indices $i' \in I'$ as follows.

Let $i' \in I'$, and set $i = f(i')$. Suppose $q_{i'} := q_i : s_1 \times \cdots \times s_n \to s$. Because f is saturated, there are unique elements $a_{i'}(1), \dots, a_{i'}(n) \in Y$ such that $f(a_{i'}(1)) = a_i(1), \dots, f(a_{i'}(n)) = a_i(n)$ and $p_Y(a_{i'}(1), \dots, a_{i'}(n)) = i'$. By definition of p_Y, we have $a_i(1), \dots, a_i(n) \in X$, and hence $a_{i'}(1), \dots, a_{i'}(n) \in Y' \times_Y X$. The $b_{i'}$ are then due to $p_Y(a_{i'}(1), \dots, a_{i'}(n)) = i'$.

Clearly

$$p_{Y'}(y'_1, \dots, y'_n) = p_{Y' \times_Y X}(y'_1, \dots, y'_n) \tag{3.6}$$

for all operation symbols $p : s_1 \times \cdots \times s_n \to s$ and $y'_1, \dots, y'_n \in \operatorname{dom} p_{Y' \times_Y X}$, and

$$p_{Y'}(a_{i'}(1), \dots, a_{i'}(n)) = i' \tag{3.7}$$

for all $i' \in I'$.

We claim that the partial functions $p_{Y'}$ are fully described by (3.6) and (3.7). Let $y'_1, \dots, y'_n \in Y$ and $p_{Y'}(y'_1, \dots, y'_n) = y'$ for some y' and operation symbol p. If $f(y') \in X$, then $y' \in Y' \times_Y X$ and so (3.6) applies. Otherwise, $f(y') = i$ for some $i \in I$ and $p = q_i$. Thus, $i' := y' \in I'$. But then $y'_1 = a_{i'}(1), \dots, y'_n = a_{i'}(n)$ because f is saturated, and (3.7) applies.

By 3.2.6, (3.6) is a pushout square, so we conclude that t' is totalizing. □

Corollary 3.2.13. *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *be a pair of composable morphisms in* Palg(Σ) *such that* gf *is totalizing and* g *is a monomorphism and saturated. Then* f *is totalizing.*

Proof. Because g is a monomorphism,

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "f"] & X \arrow[d, "gf"] \\
Y \arrow[r, "g"] & Z
\end{tikzcd}$$

is a pullback square. We conclude with 3.2.12. □

Corollary 3.2.14. *The class of totalizing morphisms has the two-out-of-three property: If* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *is a pair of composable morphisms in* Palg(Σ) *and two of the three morphisms* f, g *and* gf *are totalizing, then so is the third.*

Proof. The case where f and g are assumed to be totalizing follows from 2.1.4, and so does the case where gf and f are assumed to be totalizing because then f is an epimorphism.

The last case where g and gf are assumed to be totalizing follows from 3.2.13 because every totalizer is saturated and a monomorphism. ☐

3.3 Partial Horn logic

In this section, Palmgren and Vicker's *partial Horn logic* [17] is interpreted as a convenient and copmlete tool for the description of epimorphisms of finite partial Σ-algebras. The models of such partial Horn logic theories are then precisely the Σ-algebras orthogonal to the corresponding epimorphisms, from which we immediately obtain the existence of free models. Thus, our approach does not rely on any inference system (unlike [17]) and instead uses the formalism of orthogonality.

After finishing the manuscript, we learned that the "completeness" theorem for partial Horn logic with respect to epimorphisms of finite partial algebras (theorem 3.3.6) has already appeared in P. Burmeister's work, ([5], lemma 9.5). There, what we will refer to as partial Horn logic sequents are called "*elementary implications* (definition 9.2 ibid.). Nevertheless, the proof based on totalizing morphisms as presented here is to the author's knowledge novel.

Definition 3.3.1. Let $\Sigma = (S, P, \mathrm{ar})$ be a signature. Let $V \in \mathrm{Palg}(\Sigma)$ be a finite S-sorted set. A *term* over Σ in the variables V is an element of V^{tot}, where $V \to V^{\mathrm{tot}}$ is the totalization of V.

Proposition 3.3.2. *Let $\Sigma = (S, P, \mathrm{ar})$ be a signature. Let V be a finite S-sorted set, and let $Q \subseteq V^{\mathrm{tot}}$ be a set of terms in the variables V. Then there is a least partial subalgebra $\langle Q\rangle \hookrightarrow V^{\mathrm{tot}}$ of V^{tot} containing V such that $V \to \langle Q\rangle$ is totalizing. If T is finite, then $\langle T\rangle$ is finite.*

Proof. Consider the set I of all saturated partial subalgebras $X_i \subseteq V^{\mathrm{tot}}$ such that $T \subseteq X_i$. The inclusion of their intersection $\langle Q\rangle := \bigcap_i X_i \subseteq V^{\mathrm{tot}}$ (i.e. their product in $\mathrm{Palg}(\Sigma)_{/V^{\mathrm{tot}}}$) is again saturated because the saturated morphisms are a right-orthogonality class and hence closed under limits. Then by 3.2.13, $V \to \langle Q\rangle$ is totalizing. If $V \to K \to V^{\mathrm{tot}}$ is an arbitrary partial subalgebra of V^{tot} containing V such that $V \to K$ is totalizing, then by the two-out-of-three property, $K \to V^{\mathrm{tot}}$ is totalizing and in particular saturated. It follows that $\langle Q\rangle$ is contained in K.

Now suppose that T is finite. Because

$$\begin{tikzcd}
V \arrow[r] \arrow[d] & \langle\{q\}\rangle \arrow[d] \\
\langle Q\rangle \arrow[r] & \langle Q \cup \{q\}\rangle
\end{tikzcd}$$

is a pushout square for all (sets of) terms q, Q, we immediately reduce to the case of a singleton set $Q = \langle\{q\}\rangle$ as finite partial Σ-algebras are stable under finite colimits. This we prove by induction, i.e. using the universal property of V^{tot}. (Technically, the following paragraph is a description of a total algebra whose elements are certain finite partial Σ-algebras, a morphism of V into it and the induced morphism from V^{tot} into it assigns each term q the partial Σ-algebra $\langle\{q\}\rangle$.)

Clearly $\langle\{q\}\rangle = V$ if $q \in V$. Now let $q = p(q_1, \dots, q_n)$ and suppose that $\langle\{q_i\}\rangle$ is finite for all $i \in \{1, \dots, n\}$. Let $Q = \{q_1, \dots, q_n\}$.

$$\langle Q\rangle = \langle\{q_1\}\rangle \amalg_V \cdots \amalg_V \langle\{q_n\}\rangle$$

is finite. Let K be the finite partial Σ-algebra given by $\operatorname{car} K = (\operatorname{car}\langle Q\rangle) \cup \{q\}$ and partial functions p_K inherited from $\langle Q\rangle$ or given by $p_K(q_1, \dots q_n) = q$. The inclusion $\langle Q\rangle \to K$ is totalizing, and thus so is $V \to K$. It follows that $\langle q\rangle \subseteq K$, and the other inclusion follows because $\langle\{q\}\rangle \to V^{\text{tot}}$ is totalizing and in particular saturated. ☐

Definition 3.3.3. Let Σ be a signature. We define (quasi-equational) *partial Horn logic* over Σ. The *formulae* over Σ with variables in some finite S-sorted set V are freely generated according to the following clauses.

- $\top$ is a formula.
- If q_1 and q_2 are terms over Σ in the variables V and q_1 and q_2 have the same sort, then $q_1 = q_2$ is a formula.
- If ϕ and ψ are formulae, then $\phi \wedge \psi$ is a formula.

If ϕ and ψ are formulae with variables in V, then $\phi \vdash \psi$ is a *sequent* with variables in V.

Formally, the formulae of partial Horn logic can be defined using the apparatus of totalizing morphisms as follows. We extend Σ to a signature $\Sigma' = (S', P', \mathrm{ar}')$. Σ' has an additional sort $f \in S'$, the sort of formulae. For each $s \in S$ we add operations $- =_s - : s \times s \to f$, and operations $- \wedge - : f \times f \to f$ and a constant $\top : f$ corresponding to finite conjunctions. We treat an S-sorted set V as an S'-sorted set with $V_f = \emptyset$ via the left adjoint F^* arising from the inclusion $F : \Sigma \hookrightarrow \Sigma'$. Then formulae with variables in V are exactly the elements $(F^*(V)^{\text{tot}})_f$, where $F^*(V) \to F^*(V)^{\text{tot}}$ is a totalization in $\mathrm{Palg}(\Sigma')$.

The sets of variables for formulae or sequents will usually not be mentioned explicitly. Unless specified otherwise, the variables are those that occur, with sorts determined by usage. Note that a sequent $\phi \vdash \psi$ is well-formed only if ϕ and ψ have the same variables. If left implicit, the set of variables of $\phi \vdash \psi$ are thus given by the variables occuring in ϕ or ψ.

Analogous the prevalent notation of the type theory community, we will also use an alternative syntax in 4 By a juxtaposition

$$\phi \qquad \psi$$

of formulae ϕ, ψ, we mean the conjunction $\phi \wedge \psi$, associated to the left if there are more than two formulae. A sequent $\phi \vdash \psi$ is alternatively written as

$$\frac{\phi}{\psi}$$

i.e. premise and conclusion separated by a horizontal line. If the premise ϕ is the truth symbol $\top$, we write

$$\frac{}{\psi}$$

For each term q, we abbreviate

$$t \downarrow \equiv t = t$$

(here, the triple equality sign $\equiv$ is meta-theoretical and used to distinguish it from the normal equality $=$ as used in partial Horn logic); it can be read as "t is defined".

When these simplifications will become relevant, we are interested in defining whole *sets* of sequents. In this context, we write

$$\frac{\phi}{\overline{\psi}}$$

and mean the two-element set given by the sequents $\phi \vdash \psi$ and $\psi \vdash \phi$.

Concrete examples can be found in section 4.

Definition 3.3.4. Let $\Sigma = (S, P, \mathrm{ar})$ be a signature. To each formula ϕ over Σ with variables in some finite S-sorted set V, we associate a finite partial Σ-algebra $\langle \phi \rangle$ together with a canonical morphism $V \to \langle \phi \rangle$ according to the following clauses.

- $\langle \top \rangle = V$
- $\langle q_1 = q_2 \rangle = \langle \{q_1, q_2\} \rangle / R$ where $R = \{(q_1, q_2)\}$
- $\langle \phi \wedge \psi \rangle = \langle \phi \rangle \amalg_V \langle \psi \rangle$.

If $\phi \vdash \psi$ is a sequent in the variables V, then $\langle \phi \vdash \psi \rangle : \langle \phi \rangle \to \langle \phi \wedge \psi \rangle$ is defined by the pushout square

$$\begin{tikzcd}
V \arrow[r] \arrow[d] & \langle \psi \rangle \arrow[d] \\
\langle \phi \rangle \arrow[r] & \langle \phi \wedge \psi \rangle.
\end{tikzcd}$$

Definition 3.3.5. A *theory* is a tuple $(\Sigma, \mathbb{T})$, where Σ is a formula and $\mathbb{T}$ is a set of sequents over Σ. (The sets of variables may be distinct for different sequents in $\mathbb{T}$). We define

$$\langle (\Sigma, \mathbb{T}) \rangle = \{ \langle t \rangle \mid t \in \mathbb{T} \};$$

this is a set of morphisms in $\mathrm{Palg}(\Sigma)$. A sequent $\phi \vdash \psi$ is an *axiom* of $(\Sigma, \mathbb{T})$ if it is an element of $\mathbb{T}$; if

$$\langle \phi \vdash \psi \rangle \in {}^{\perp}(\langle \mathbb{T} \rangle^{\perp}),$$

it is *admissible.* A partial Σ-algebra X is a $(\Sigma, \mathbb{T})$*-model* if $!_X \in \langle \mathbb{T} \rangle^{\perp}$, where $!_X : X \to \top$ is the unique morphism to the terminal partial Σ-algebra $\top$. We denote the full subcategory of $\mathrm{Palg}(\Sigma)$ given by the $\mathbb{T}$-models by $\mathrm{Mod}(\mathbb{T})$.

Unless confusion is possible, we will identify a theory $(\Sigma, \mathbb{T})$ with the corresponding set of sequents $\mathbb{T}$.

The intuition for the semantics is as follows. First of all, asserting the orthogonality $\langle \phi \vdash \psi \rangle \perp !_X$ amounts to asserting that the lifting problem

$$\begin{tikzcd}
\langle \phi \rangle \arrow[r, "a"] \arrow[d, "\langle \phi \vdash \psi \rangle"'] & X \\
\langle \phi \wedge \psi \rangle \arrow[ur, dashed, "\exists b"'] &
\end{tikzcd}$$

has a unique solution b for all a.

It will be proved below that $\langle \phi \vdash \psi \rangle$ is an epimorphism. Thus, uniqueness of a solution is automatic. Let V be the set of variables for $\phi \vdash \psi$. Then a map a as above is given by elements of the corresponding sorts for each of the variables in V such that the terms occuring in ϕ are defined after subsituting the variables and satisfy the equalities asserted by ϕ. The extension b along $\langle \phi \vdash \psi \rangle$ exists if the terms occuring in b are, after appropriate subsitution, defined as well and satisfy the equations of ψ.

There is no essential difference between sets of sequents and sets of epimorphisms of finite partial Σ-algebras $\mathrm{Palg}(\Sigma)$ as the following proposition shows. Thus, partial Horn logic can be understood as a convenient syntax for defining epimorphisms of finite partial Σ-algebras.

Theorem 3.3.6. *Let Σ be a signature. If $\phi \vdash \psi$ is a sequent over Σ, then $\langle \phi \vdash \psi \rangle$ is an epimorphism of finite partial Σ-algebras. Conversely, if $m : A \to B$ is an epimorphism of finite partial Σ-algebras, then there are formulae ϕ, ψ such that $m \cong \langle \phi \vdash \psi \rangle$ in the arrow category* $\mathrm{Palg}(\Sigma)^{\to}$*.*

Proof. Let V be a finite S-sorted set. By induction, it is easy to see that $V \to \langle \phi \rangle$ is an epimorphism of finite partial Σ-algebras for all formulae ϕ with variables in V. Then pushouts of such morphisms are epimorphisms, in particular $\langle \phi \vdash \psi \rangle$ for all formulae ψ with variables in V.

Let A be finite partial Σ-algebra. Let $V = \operatorname{car} A$ and let ϕ be the formula with variables in V given by

$$\phi \equiv \bigwedge_{\substack{p \\ x_1,\dots,x_n \in A}} p(x_1, \dots, x_n) = p_A(x_1, \dots, x_n),$$

where $p(x_1, \dots, x_n) = p_{V^{\mathrm{tot}}}(x_1, \dots, x_n)$ denotes a term, i.e. an element of a totalization V^{tot} of V. Then clearly the map $V = \operatorname{car} A \to A$ factors via $\langle \phi \rangle$ and induces an isomorphism of partial Σ-algebras.

Now let furthermore $m : A \to B$ be an epimorphism of partial Σ-algebras with B finite. Because total and totalizing morphisms constitute an factorization system, we find a factorization

$$\begin{tikzcd}
 & & A' \arrow[dr, "a"] & \\
\operatorname{car} A \arrow[urr, "t"] \arrow[rr] & & A \arrow[r, "m"] & B
\end{tikzcd}$$

with a total and t totalizing. Because the composite $\operatorname{car} A \to A \xrightarrow{m} B$ is an epimorphism, a is an epimorphism and hence, because it is also total, an effective epimorphism by 3.2.3. In particular, a_s is surjective for all s. Set

$$A_0 = \langle \{x_y \mid y \in Y\} \rangle,$$

where the $x_y \in A'$ are choices such that $a(x_y) = y$ for all (finitely many) $y \in B$. Thus, A_0 is the least partial subalgebra of A' containing $\operatorname{car} A$ such that the inclusion $\operatorname{car} A \hookrightarrow A_0$ is totalizing. Clearly the components $(a_0)_s$ of the restriction $a_0 : A_0 \hookrightarrow A' \xrightarrow{a} B$ of a are surjective for all s.

By finiteness of A_0 and B, the set of all commutative squares of the form

$$\begin{tikzcd}
A^q \arrow[r] \arrow[d, "m^q"] & A_0 \arrow[d, "a_0"] \\
B^q \arrow[r] & B
\end{tikzcd}$$

for some q is finite. It follows that that the partial Σ-algebra A_1' arising from an M-extending square

$$\begin{tikzcd}
A_0 \arrow[r, hook, "u"] \arrow[d, "a_0"] & A_1 \arrow[d, "a_1"] \\
B \arrow[r, equal] & B
\end{tikzcd} \tag{3.8}$$

with u totalizing as constructed in 2.2.5 *(i)* is a finite colimit of finite partial Σ-algebras and hence itself finite.

Because $(a_0)_s$ is surjective, $(a_1)_s$ is surjective for all s. If $x_1, \dots, x_n \in \operatorname{dom} p_B$ for some p, there are preimages $y_1, \dots, y_n \in A_0'$ under a_0. (3.8) is M-extending and $m^p \in M$, hence

$$p_{A_1'}(l(a_1), \dots, l(a_n)) \downarrow$$

and $\operatorname{dom} p_{A'_1} \to \operatorname{dom} p_B$ is surjective. In particular, a_1 is an effective epimorphism, thus $B \cong A_1/R$ where $R = \{(x_1, x_2) \mid a_1(x_1) = a_1(x_2)\}$ is the kernel congruence of a_1.

It follows that for the formula with variables in car A

$$\psi \equiv \bigwedge_{\substack{x_1, x_2 \in A_1 \\ a_1(x_1) = a_1(x_2)}} x_1 = x_2,$$

we have

$$\langle \psi \rangle \cong B.$$

There is an epimorphism

$$\langle \phi \rangle \xrightarrow{\sim} A \overset{m}{\twoheadrightarrow} B \xrightarrow{\sim} \langle \psi \rangle$$

that commutes with the canonical maps $\operatorname{car} A \to \langle \phi \rangle$ and $\operatorname{car} A \to \langle \psi \rangle$. But then

$$\langle \phi \rangle \amalg_{\operatorname{car} A} \langle \psi \rangle \cong \langle \psi \rangle \cong B$$

and $\langle \phi \vdash \psi \rangle \cong m$, which we set out to prove. □

Proposition 3.3.7. *Let* $\mathbb{T} = (\Sigma, \mathbb{T})$ *be a theory. Then the inclusion* $\operatorname{Mod}(\mathbb{T}) \to \operatorname{Palg}(\Sigma)$ *is a right adjoint.*

Proof. By 2.1.7. □

Definition 3.3.8. Let $\mathbb{T} = (\mathbb{T}, \Sigma)$ and $\mathbb{T}' = (\mathbb{T}, \Sigma')$ be theories. A *theory morphism* $F : \mathbb{T} \to \mathbb{T}'$ is a signature morphism $\tilde{F} : \Sigma \to \Sigma'$ such that

$$\tilde{F}^*(m) \in {}^{\perp}(\langle \mathbb{T}' \rangle^{\perp})$$

for all $m \in \mathbb{T}$.

Trivial examples are given by signature inclusions $\Sigma \hookrightarrow \Sigma'$ such that $\mathbb{T} \subseteq \mathbb{T}'$ (where we identify morphisms in $\operatorname{Palg}(\Sigma)$ with their image under the full and faithful functor $\operatorname{Palg}(\Sigma) \to \operatorname{Palg}(\Sigma')$). In this situation, we also say that $\mathbb{T}'$ *extends* $\mathbb{T}$.

Proposition 3.3.9. *Let* $F : \mathbb{T} \to \mathbb{T}'$ *be a theory morphism with underlying signature morphism* $\tilde{F} : \Sigma \to \Sigma'$. *Then* $\tilde{F}_* : \operatorname{Palg}(\Sigma') \to \operatorname{Palg}(\Sigma)$ *restricts to a functor*

$$F_* : \operatorname{Mod}(\mathbb{T}') \to \operatorname{Mod}(\mathbb{T}),$$

which has a left adjoint $F^* : \operatorname{Mod}(\mathbb{T}) \to \operatorname{Mod}(\mathbb{T}')$.

Proof. Let $X \in \operatorname{Mod}(\mathbb{T}')$ and $m \in \mathbb{T}$. Then lifts

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "m"'] & \tilde{F}_* X \\
B \arrow[ur, dashed] &
\end{tikzcd}$$

are by adjointness $\tilde{F}^* \dashv \tilde{F}_*$ equivalent to lifts

$$\begin{tikzcd}
\tilde{F}^* A \arrow[r] \arrow[d, "\tilde{F}^* m"'] & X \\
\tilde{F}^* B \arrow[ur, dashed] &
\end{tikzcd}$$

which exist uniquely because $X \in \mathrm{Mod}(\mathbb{T}')$ and $\tilde{F}^*(m) \in {}^{\perp}(\langle \mathbb{T}' \rangle^{\perp})$ Thus, $F_*(X) = \tilde{F}_*(X) \in \mathrm{Mod}(\mathbb{T})$.

Denote by $U : \mathrm{Mod}(\mathbb{T}') \to \mathrm{Palg}(\Sigma')$ the inclusion functor. Thus, $F_* = \tilde{F}_* \circ U$. By 3.3.7, U has a right adjoint V. Because adjunctions compose, we have

$$V \circ \tilde{F}^* \dashv \tilde{F}_* \circ U$$

so $F^* := V \circ \tilde{F}^*$ is a left adjoint to F_*. □

3.4 Validity

As will be shown in section 4, partial Horn logic can be used to obtain syntactic presentations of free categories with various types of algebraic structure. Fix a partial Horn logic theory $\mathbb{T} = (\Sigma, \mathbb{T})$. In [17], the free $\mathbb{T}$-model over some partial Σ-algebra X is obtained as follows. First, the signature Σ is augmented by constant symbols for all elements of X. Then, the theory $\mathbb{T}$ is augmented by axioms that assert that the corresponding constants are defined, and axioms that enforce the result of applying operations p on these symbols if p_X is defined on the corresponding elements of X. Then it is shown that the free $\mathbb{T}$-model over X is given by the provably defined terms modulo provable equality; this relies crucially on a complete calculus of logical inference.

The role of a computer-based proof-checker for a partial Horn logic theory is now straightforward: The user produces a list of terms over the signature and the proof-checker decides whether they are provably defined, i.e. *valid*, and can be interpreted as elements of free $\mathbb{T}$-models.

In our terminology, “terms” are elements of the totalizaton X^{tot} of X. Then the construction of the free $\mathbb{T}$-model described above shows that it can be obtained as subquotient

$$\begin{tikzcd}
X_0 \arrow[r, hook] \arrow[d, two heads] & X^{\mathrm{tot}} \\
X_0/R &
\end{tikzcd}$$

for some partial Σ-algebra X_0 and congruence R. It is easy to see that X_0 and R as constructed in [17] have the property that $X_0 \to X_0/R$ is total. The formalism presented here has the advantage that we can exhibit X_0 and R as part of a universal construction, thereby showing that they are unique up to unique isomorphism (theorem 3.4.1).

We continue by showing that deciding validity is equivalent to deciding the domains of operations in free $\mathbb{T}$-models, and that if the axioms of a theory can be well-ordered suitably, then $\mathbb{T}$-models have decidable domains if their carrier sets have decidable equality. The theories presented in section 4 all satisfy this well-orderedness condition, although equality (and hence validity) in the respective free models is usually undecidable.

Theorem 3.4.1. *Let X be a partial Σ-algebra and let $r : X \to X'$ be its reflection into* $\mathrm{Mod}(\mathbb{T})$. *Let $t : X \to X^{\mathrm{tot}}$ be a totalization of X. There is, up to unique compatible isomorphism, a unique X_0 that fits into a commutative diagram*

$$\begin{tikzcd}
X \arrow[r, "t_0"] \arrow[rr, bend left, "t"] \arrow[dr, "r"'] & X_0 \arrow[r, "t_1"] \arrow[d, "a"] & X^{\mathrm{tot}} \\
& X' &
\end{tikzcd}$$

with t_0 totalizing and a total. t_1 is totalizing and a monomorphism.

Proof. t and a are due to the factorization system of totalizing and total morphisms. Let $t_1 : X_0 \to X_1$ be a totalization of X_0. Then because both t and $t_1 \circ t_0$ are both totalizations of X, we have $X_1 \cong X^{\mathrm{tot}}$. Thus, we may assume that $X_1 = X^{\mathrm{tot}}$. t_1 is totalizing by the two-out-of-three property and in particular a monomorphism.

Now if

$$\begin{tikzcd}
X \arrow[r, "\tilde{t}_0"] \arrow[rr, bend left, "t"] \arrow[dr, "r"'] & \tilde{X}_0 \arrow[r, "\tilde{t}_1"] \arrow[d, "\tilde{a}"] & X^{\mathrm{tot}} \\
& X' &
\end{tikzcd}$$

is another diagram with $\tilde{t}_0$ totalizing and $\tilde{a}$ total, then by orthogonality there is a unique isomorphism $\tilde{X}_0 \cong X_0$ which is compatible with $a, \tilde{a}$ and $t_0, \tilde{t}_0$. The compatibility of t_1 with $\tilde{t}_1$ follows from the fact that t_0 and $\tilde{t}_0$ are totalizing and hence epimorphisms. ☐

Definition 3.4.2. In the situation of 3.4.1, an element $q \in X^{\mathrm{tot}}$ is called *valid over* X if $q \in X_0$.

Definition 3.4.3. A partial Σ-algebra X has *decidable domains* if $\operatorname{dom} p_X \hookrightarrow X_{s_1} \times \cdots \times X_{s_n}$ is a decidable subset for all operation symbols $p : s_1 \times \cdots \times s_n \to s$.

Proposition 3.4.4. *Let $r : X \to X'$ be a reflection into* $\mathrm{Mod}(\mathbb{T})$*. Then validity over X is decidable if and only if X' has decidable domains.*

Proof. We use the notation from 3.4.1 throughout the proof.

Suppose first that validity over X is decidable. Let $p : s_1 \times \cdots \times s_n \to s$ be an operation symbol and let $x'_1, \ldots, x'_n \in X'_{s_1} \times \cdots \times X'_{s_n}$. Because a is both total and an epimorphism, it is an effective epimorphism by 3.2.3. In particular, a_s is surjective for all s. Thus, there are $x_1, \ldots, x_n \in X$ such that $a(x_i) = x'_i$ for all i. Then

$$\begin{aligned}
&p_{X'}(x'_1, \ldots, x'_n) \downarrow \\
\iff &p_{X_0}(x_1, \ldots, x_n) \downarrow \\
\iff &p_{X^{\mathrm{tot}}}(x_1, \ldots, x_n) \text{ is valid,}
\end{aligned}$$

which is decidable. (For the direction $\Leftarrow$ of the last equivalence, recall from 3.2.11 that all totalizing morphisms are saturated, in particular $t_1 : X_0 \to X^{\mathrm{tot}}$.)

Now suppose that X' has decidable domains. Define a (total) algebra Z by

$$Z_s = X'_s \amalg \{*\}$$

for each $s \in S$ and some singleton set $\{*\}$ and

$$p_Z(z_1, \ldots, z_n) = \begin{cases} p_{X'}(z_1, \ldots, z_n) \in X' \subseteq Z & \text{if } z_1, \ldots, z_n \in \operatorname{dom} p_{X^{\mathcal{T}}} \\ * & \text{otherwise.} \end{cases}$$

for each operation $p : s_1 \times \cdots \times s_n \to s$. Note that the case distinction is valid by the assumption that X' has decidable domains. Denote by $i : X^{\mathcal{T}} \hookrightarrow Z$ be the canonical inclusion. Clearly i_s is the inclusion of a decidable subset for each s.

Let $t : X \to X^{\mathrm{tot}}$ be a totalization of X. The composite morphism $X \xrightarrow{r} X' \xrightarrow{i} Z$ induces via the universal property of X' a morphism $f : X^{\mathrm{tot}} \to Z$ such that

$$\begin{tikzcd}
X \arrow[r, "t"] \arrow[d, "r"] & X^{\mathrm{tot}} \arrow[d, "f"] \\
X' \arrow[r, "i"] & Z
\end{tikzcd}$$

commutes. Taking the fibre product of the lower right cospan, we obtain a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "t_0"] \arrow[rr, bend left, "t"] \arrow[dr, "r"'] & X_0 \arrow[r, "t_1"] \arrow[d, "a"] & X^{\mathrm{tot}} \arrow[d, "f"] \\
 & X' \arrow[r, "i"] & Z.
\end{tikzcd}$$

where $Y = X^{\mathcal{T}} \times_Z X^{\mathrm{tot}}$.

X^{tot} is a total algebra, so trivially f is total. Because total morphisms are stable under pullback, a is total. Clearly $i : X' \to Z$ is saturated and a monomorphism. Both properties are stable under pullback, so t_1 is saturated and a monomorphism. We conclude that t_0 is totalizing by 3.2.13.

Decidable subsets are preserved by pullbacks in Set. Pullbacks in $\mathrm{Palg}(\Sigma)$ are computed as pullbacks in a functor category and hence componentwise as pullback in Set. It follows that $(t_1)_s$ is the inclusion of a decidable subset for each s. ☐

Suppose $(\prec) \subseteq P \times P$ is a well-ordering of the operation symbols P. For each $p \in P$, define the restriction of Σ to operations $\prec p$ as $\Sigma^{\prec p} = (S, P^{\prec p}, \mathrm{ar}^{\prec p})$, where

$$P^{\prec p} = \{p' \in P \mid p' \prec p\}$$

and $\mathrm{ar}^{\prec p} = \mathrm{ar}\,|_{P^{\prec p}}$ is the restriction of ar to $P^{\prec p}$. Thus, $\Sigma^{\prec p}$ has the same sorts as Σ but only $\prec$-smaller operation symbols. We identify all partial $\Sigma^{\prec p}$-algebras Y with the obvious partial Σ-algebras with entirely undefined partial functions p' if $p' \not\prec p$.

For each operation symbol $p : s_1 \times \cdots \times s_n \to s$ of Σ, let V^p be the S-sorted set given by distinct symbols v_i of sort s_i for $i = 1, \ldots, n$, and abbreviate

$$p(v_1, \ldots, v_n) \downarrow \equiv p(v_1, \ldots, v_n) = p(v_1, \ldots, v_n).$$

Proposition 3.4.5. *Suppose $(\prec)$ is a well-ordering of the operation symbols P such that for all operation symbols $p : s_1 \times \ldots s_n \to s$, there is a formula ϕ in partial Horn logic over the signature $\Sigma^{\prec p}$ with variables in V such that the two sequents*

$$\frac{\phi}{p(v_1, \ldots, v_n) \downarrow} \qquad\qquad \frac{p(v_1, \ldots, v_n) \downarrow}{\phi}$$

are admissible.

Suppose X is a $\mathbb{T}$-model such that such that X_s is a decidable set for all $s \in S$. Then X has decidable domains.

Proof. Because $(\prec)$ is a well-ordering, it suffices to prove that

$$\operatorname{dom} p'_X \text{ is decidable for all } p' \prec p \implies \operatorname{dom} p_X \text{ is decidable}$$

for all operation symbols $p : s_1 \times \cdots \times s_n \to s$.

Let $(x_1, \ldots, x_n) \in X(s_1) \times \cdots \times X(s_n)$. We have an obvious morphism $V_p \to X$ that sends v_i to x_i. Then $p(x_1, \ldots, x_n) \downarrow$ if and only if $V_p \to X$ factors via $f : V_p \to \langle \phi \rangle$.

It follows from the clauses of 3.3.4 that $\langle \phi \rangle = \langle Q \rangle / R$, where Q is the (finite) set of terms occuring in ϕ and R is a congruence. Thus, $p(x_1, \ldots, x_n) \downarrow$ if and only if $V_p \to X$ factors via $\langle Q \rangle$ and the corresponding morphism $\langle Q \rangle \to X$ identifies elements related by R.

As Q is finite and contains terms mentioning only operations $p' \prec p$, the former can be decided using the induction hypothesis. If this is the case, then checking whether elements related to by R are are identified after mapping into X is a matter of deciding equality of elements in X_s for various s, which is possible by assumption. ☐

4 Categories with algebraic structure

In this section, categories with various types of additional algebraic structure are exhibited as models of partial Horn logic theories: Categories with no additional structure, left exact (= finitely complete) categories, locally cartesian closed (lcc) categories and toposes. In each case, we also consider theories of corresponding *sketches*.

Theories come with a notion of morphism, namely the morphism of partial algebras. In the case of categories with no algebraic structure or property, these are precisely functors. However, in the other cases, this notion is too restrictive. For example, we will see that the category of models of the theory of left exact categories sLex is the category of *strict* left exact categories, which has *strict* left exact functors as morphisms: While left exact functors $\mathcal{C} \to \mathcal{D}$ need only preserve finite limits up to isomorphism, *strict* left exact functors have to map the canonical choices for pullback squares and terminal object in $\mathcal{C}$ to the canonical choices in $\mathcal{D}$.

The theories for the four types of algebraic structure mentioned above extend one another naturally. For example, the theory of left exact categories is an extension of the theory of categories, and in each case of algebraic structure, the theory of categories with the respective structure extends the theory of sketches for this structure. These extensions give rise to adjoint pairs of functors, so that we obtain e.g. free categories over linear sketches or free strict left exact categories over categories.

Unfortunately, the arising free categories with algebraic structure are free only among the corresponding *strict* functors, i.e. the functors preserving assigned algebraic structure on the nose. Consider the free (strict) left exact category $\mathcal{C}$ over the empty category. There are two distinct constant functors $\mathcal{C} \to \mathcal{I}$ to the free standing isomorphism $\mathcal{I}$ and both preserve finite limits up to isomorphism, i.e. are left exact. It follows that $\mathcal{C}$ is not a free object in the category of left exact categories and left exact functors, and indeed there is no initial left exact category.

Luckily, categories of categories with algebraic structure have canonical structures as 2-categories, with either general natural transformations or natural isomorphisms as 2-cells. We will then see that the left adjoints arising from theory extensions mentioned above give rise to left *biadjoints*, i.e. bifree categories with additional structure. Thus, if $\mathcal{C}$ is the bifree left exact category over the empty category, there are many left exact functors $\mathcal{C} \to \mathcal{I}$ to the interval, but every two such functors are uniquely isomorphic.

The existence of the (bi)adjunctions constructed in this section also follows from general 2-dimensional monad theory; see [18] for an overview, [13] for sketches and compare proposition 3.1 and theorems 3.9 and 5.1 of [3] to the results of 4.2. While the results of this section are just special cases of 2-dimensional monad theory, the advantage of the approach presented here is that we obtain concrete syntactical presentations of (bi)free categories with algebraic structure. Furthermore, the sketches of [13] always come with an underlying category, whereas the ones we consider are based on *linear sketches*. The advantage of linear sketches instead of categories is that finitely presentable categories (possibly with additional algebraic structure) can be presented by finite linear sketches. For example, a free-standing endomorphism $f : x \to x$ is a finite linear sketch with one morphism and one object, but the free category (left exact category, lcc category, ...) over this datum is infinite. In [12] D2, sketches given by directed graphs and additional data are considered, but only limit-colimit sketches are considered.

4.1 2-categorical prerequisites

Some previous exposure to higher categorical concepts will be assumed throughout section 4. However, we introduce the notation and nomenclature used here. We suggest [12] or [8] as references.

A *2-category* $\mathcal{A}$ is a Cat-enriched category. In other words, $\mathcal{A}$ consists of

- a set $\operatorname{Ob}\mathcal{A}$;
- categories $\mathcal{A}(x,y)$ for all objects $x, y \in \operatorname{Ob}\mathcal{A}$;
- objects $\mathrm{id}_x \in \mathcal{A}(x,x)$ for all $x \in \operatorname{Ob}\mathcal{A}$; and
- functors $(- \circ -) : \mathcal{A}(y,z) \times \mathcal{A}(x,y) \to \mathcal{A}(x,z)$ for all $x, yz \in \operatorname{Ob}\mathcal{A}$.

The composition functors and identity objects have to satisfy the obvious identity and associativity laws on the nose. We refer to elements $x, y \in \operatorname{Ob}\mathcal{A}$ as objects, to elements $f, g \in \operatorname{Ob}\mathcal{A}(x,y)$ as morphisms and to elements $\alpha \in \mathcal{A}(x,y)(f,g)$ as 2-cells. In this situation, we write $f : x \to y$, $g : x \to y$ and $\alpha : f \Rightarrow g$. Note that if α and β are 2-cells, then the $\circ$ operator in $\beta \circ \alpha$ can refer to different kinds of composition, depending on the signature of α and β: In the situation

$$\begin{tikzcd}
x \arrow[rr, "f"] & {} \arrow[d, Rightarrow, "\alpha"] & y \\
x \arrow[rr, "g" description] & {} \arrow[d, Rightarrow, "\beta"] & y \\
x \arrow[rr, "h"'] & {} & y
\end{tikzcd}$$

$\beta \circ \alpha : f \Rightarrow h$ is the *vertical* composition, i.e. composition in the category $\mathcal{A}(x,y)$. If we have

$$\begin{tikzcd}
x \arrow[rr, "f_1"] & {} \arrow[d, Rightarrow, "\alpha"] & y \arrow[rr, "g_1"] & {} \arrow[d, Rightarrow, "\beta"] & z \\
x \arrow[rr, "f_2"'] & {} & y \arrow[rr, "g_2"'] & {} & z
\end{tikzcd}$$

then the composition operator $\circ$ in

$$\beta \circ \alpha : g_1 \circ f_1 \Rightarrow g_2 \circ f_2$$

refers to the action of the composition functor $\mathcal{A}(x,y) \times \mathcal{A}(y,z) \to \mathcal{A}(x,z)$ on objects and morphisms, the *horizontal* composition. If $g_1 = g_2 = g$ and $\beta = \mathrm{id}_g : g \Rightarrow g$ is the identity 2-cell, then $g \circ \alpha$ refers to the horizontal composition $\mathrm{id}_g \circ \alpha$ with $\mathrm{id}_g : g \Rightarrow g$, and similarly for $\beta \circ f$ if $f = f_1 = f_2$. As usual, we will often suppress the composition operator and write gf and $\beta\alpha$ instead of $g \circ f$ and $\beta \circ \alpha$.

Let $\mathcal{A}, \mathcal{B}$ be 2-categories. A *2-functor* $F : \mathcal{A} \to \mathcal{B}$ consists of a map $\operatorname{Ob} F : \operatorname{Ob}\mathcal{A} \to \operatorname{Ob}\mathcal{B}$ and functors $F = F_{x,y} : \mathcal{A}(x,y) \to \mathcal{B}(Fx, Fy)$ for all $x, y \in \operatorname{Ob}\mathcal{A}$ which have to be compatible with the identity and composition functors on the nose. 2-functors are composed in the obvious way and there are identity 2-functors $\mathrm{Id}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ on every 2-category $\mathcal{A}$.

Let $G : \mathcal{B} \to \mathcal{A}$ be a 2-functor and let $x \in \operatorname{Ob}\mathcal{A}$. A *biuniversal morphism* from x to G consists of a an object $r \in \operatorname{Ob}\mathcal{B}$ and a morphisms $u : x \to Fr$ such that for every $y \in \operatorname{Ob}\mathcal{B}$, the functor

$$\phi : \begin{cases} \mathcal{B}(r,y) \to \mathcal{A}(x, Gy) \\ f \mapsto G(f) \circ u, & f \in \operatorname{Ob}\mathcal{B}(r,y) \\ \alpha \mapsto G(\alpha) \circ u, & \alpha \in \mathcal{B}(r,y)(f,g) \end{cases} \tag{4.1}$$

is an equivalence of categories. (Technically, an inverse to ϕ is part of the data of a biuniversal morphism.) If ϕ is an isomorphism of categories (as opposed to just an equivalence) for all y, then u will be called a *2-universal morphism.* Similarly to the 1-categorical case, a choice of 2-universal morphisms $\eta_x : x \to G(y)$ for each $x \in \operatorname{Ob}\mathcal{A}$ induces a unique 2-functor $F : \mathcal{A} \to \mathcal{B}$ such that $\eta : \mathrm{Id} \to GF$ is a natural transformation; it is the unit of a 2-adjunction $F \dashv G$. (If η_x is only biuniversal, then F is in general not a 2-functor but a bifunctor, but this need not concern us in this exposition.)

Let $F, G : \mathcal{A} \rightrightarrows \mathcal{B}$ be a pair of parallel 2-functors. A *2-natural transformation* $\phi : F \Rightarrow G$ from F to G is a family of morphisms $\phi_x : F(x) \to G(x)$ in $\mathcal{D}$ such that

$$\begin{tikzcd}
\mathcal{C}(x,y) \arrow[r, "G"] \arrow[d, "F"] & \mathcal{D}(G(x),G(y)) \arrow[d, "(\phi_x \circ -)"] \\
\mathcal{D}(F(x),F(y)) \arrow[r, "(- \circ \phi_y)"] & \mathcal{D}(F(x),G(y))
\end{tikzcd}$$

is a commutative square of functors.

Now let $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$ be 2-functors and $\eta : \mathrm{Id}_{\mathcal{A}} \Rightarrow GF$ be a 2-natural transformation. We say that η is the unit of a *biadjunction* $F \dashv G$ if for each $x \in \mathcal{A}$, the morphism $\eta_x : x \to G(F(x))$ is a biuniversal morphism from x to G. If η_x is a 2-universal morphism for all x, then $F \dashv G$ is a *2-adjunction.*

Let $f, g : x \rightrightarrows y$ be a parallel pair of morphisms in a 2-category $\mathcal{A}$. A *strong inserter* from f to g consists of a morphism $p : [f \Rightarrow g] \to x$ with target x and a 2-cell $\alpha : fp \Rightarrow gp$ which is universal in the following sense: Given any morphism $q : z \to x$ and 2-cell $\beta : fq \Rightarrow gq$, there is a unique morphism $[\beta] : z \to [f \Rightarrow g]$ such that $q = p[\beta]$ and $\beta = \alpha[\beta]$. A 2-functor $F : \mathcal{A} \to \mathcal{B}$ is said to *preserve the strong inserter* p, α if $F(p, F(\alpha)$ is a strong inserter from $F(f)$ to $F(g)$.

Similarly, a *strong iso-inserter* from f to g consists of a universal pair of morphism $p : [f \overset{\sim}{\Rightarrow} g]$ and a 2-isomorphism (i.e. invertible 2-cell) $\alpha : fp \overset{\sim}{\Rightarrow} gp$.

Discarding 2-cells, we associate to every 2-category $\mathcal{A}$ its underlying 1-category $\mathcal{A}_\circ$, to every 2-functor F a 1-functor $F_\circ : \mathcal{A}_\circ \to \mathcal{B}_\circ$ and to every 2-natural transformation $\alpha : F \Rightarrow G$ a 1-natural transformation $\alpha_\circ : F_\circ \Rightarrow G_\circ$.

4.2 Some results on biadjunctions

We prove key lemmas which will be used to extend 1-categorical adjunctions higher adjunctions. While the author's inspiration for the formulation of these lemmas was the proof of lemma 55 in [17], he learned later that very similar lemmas already appeared in [3].

Lemma 4.2.1. *Let $G : \mathcal{B} \to \mathcal{A}$ be 2-functor and let $\tilde{F} : \mathcal{A}_\circ \to \mathcal{B}_\circ$ be left adjoint to $G_\circ$, with unit $\tilde{\eta} : \mathrm{Id}_\circ \Rightarrow G_\circ \tilde{F}$. Suppose that G is full and faithful on 2-cells and that the unit $\tilde{\eta}_x : x \to G(F(x))$ is an isomorphism for all $x \in \operatorname{Ob}\mathcal{A}$. Then $\tilde{F}$ can be extended to a 2-functor F and $\tilde{\eta}$ can be extended to a 2-natural transformation $\eta : \mathrm{Id} \Rightarrow GF$ which is the unit of a 2-adjunction $F \dashv G$.*

Proof. We prove that $\tilde{\eta}_x$ is 2-universal for all $x \in \operatorname{Ob}\mathcal{A}$. The functor ϕ as in (4.1) is an isomorphism on objects by the 1-categorical universal property of $\tilde{\eta}_x$. Let

$$\begin{tikzcd}
x \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "\tilde{\eta}_x"] & G(y) \\
G(F(x)) \arrow[ur, shift left, "G(\bar{f})"] \arrow[ur, shift right, "G(\bar{g})"'] &
\end{tikzcd}$$

be a diagram in $\mathcal{A}$ such that $f = \eta_x \circ G(\bar{f})$ and $g = \eta_x \circ G(\bar{g})$. We have to show that a 2-cell $\alpha : f \Rightarrow g$ can be uniquely extended to a natural transformation $\bar{\alpha} : \bar{f} \to \bar{g}$ such that

$$G(\bar{\alpha})\tilde{\eta}_x = \alpha. \tag{4.2}$$

η_x is invertible and G is full and faithful on 2-cells, so

$$\alpha\tilde{\eta}_x^{-1} : G(\bar{f}) \to G(\bar{g})$$

is well-defined and has a unique preimage; clearly it is the unique $\bar{\alpha}$ satisfying (4.2). ☐

Lemma 4.2.2. *Let $G : \mathcal{B} \to \mathcal{A}$ be a 2-functor with left biadjoint $F : \mathcal{A} \to \mathcal{B}$. Suppose that G factors as $\mathcal{B} \xrightarrow{G^0} \mathcal{B}' \xrightarrow{G^1} \mathcal{A}$ with G^1 full and faithfull on morphisms and 2-cells. Then there is a biadjunction $G^0 \dashv FG^1$. If $F \dashv G$ is a 2-adjunction, then $G^0 \dashv FG^1$ is a 2-adjunction.*

Proof. For all $x \in \operatorname{Ob}\mathcal{B}'$ and $y \in \operatorname{Ob}\mathcal{B}$

$$\mathcal{B}'(x, G^0(y)) \cong \mathcal{B}(G^1(x), G(y)) \simeq \mathcal{A}(F(G^1(x)), y),$$

thus $G^0 \dashv FG^1$ is a biadjunction. For each $x \in \operatorname{Ob}\mathcal{B}'$, the unit $G^1(x) \to G(F(G^1(x)))$ of the biadjunction $F \dashv G$ has a unique preimage $x \to G^0(F(G^1(x))$ under G^1, which is the unit of the biadjunction $G^0 \dashv FG^1$.

If $F \dashv G$ is a 2-adjunction, then the equivalence $\mathcal{B}(G^1(x), G(y)) \simeq \mathcal{A}(F(G^1(x)), y)$ is an isomorphism and hence $G^0 \dashv FG^1$ is a 2-adjunction. ☐

Lemma 4.2.3. *Let $G : \mathcal{B} \to \mathcal{A}$ be 2-functor and let $\tilde{F} : \mathcal{A}_\circ \to \mathcal{B}_\circ$ be left adjoint to $G_\circ$ with unit $\tilde{\eta} : \mathrm{Id} \Rightarrow G_\circ\tilde{F}$. Suppose that $\mathcal{B}$ admits strong inserters to all parallel pairs of morphisms and that G preserves strong inserters. Then $\tilde{F}$ can be extended to a 2-functor F and $\tilde{\eta}$ can be extended to a 2-natural transformation $\eta : \mathrm{Id} \Rightarrow GF$ which is the unit of a 2-adjunction $F \dashv G$.*

Proof. We prove that $\eta_x = \tilde{\eta}_x$ is 2-universal for all $x \in \operatorname{Ob}\mathcal{A}$. The functor ϕ as in (4.1) is an isomorphism on objects by the 1-categorical universal property of $\tilde{\eta}_x$. Let

$$\begin{tikzcd}
x \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "\eta_x"'] & G(y) \\
G(F(x)) \arrow[ur, shift left, "G(\bar{f})"] \arrow[ur, shift right, "G(\bar{g})"'] &
\end{tikzcd}$$

be a diagram in $\mathcal{A}$ such that $f = \eta_x \circ G(\bar{f})$ and $g = \eta_x \circ G(\bar{g})$. Let $p : [\bar{f} \Rightarrow \bar{g}] \to F(x)$ be a strong inserter from f to g in $\mathcal{B}$. By assumption, $G(p) : G([G(\bar{f}) \Rightarrow \bar{g}]) \to G(F(x))$ is a strong inserter from $G(\bar{f})$ to $G(\bar{g})$ in $\mathcal{A}$. Using this and the 1-categorical universal property of $\tilde{\eta}_x$, we obtain isomorphisms

$$\begin{aligned}
&\{\bar{\alpha} : \bar{f} \Rightarrow \bar{g}\} \\
\cong &\{[\bar{\alpha}] : F(x) \to [\bar{f} \Rightarrow \bar{g}] \mid p \circ [\bar{\alpha}] = \mathrm{id}_{F(x)}\} \\
\cong &\{[\alpha] : x \to G([\bar{f} \Rightarrow \bar{g}]) \mid G(p) \circ [\alpha] = \mathrm{id}_{G(F(x))}\} \\
\cong &\{\alpha : \underbrace{G(\bar{f}) \circ \eta_x}_{=f} \Rightarrow \underbrace{G(\bar{g}) \circ \eta_x}_{=g}\}
\end{aligned}$$

and hence that ϕ from (4.1) is also full and faithful, i.e. an isomorphism. ☐

Let $G : \mathcal{B} \to \mathcal{A}$ be a 2-functor. We define the 2-category $\mathcal{B}_G$ by

$$\operatorname{Ob}\mathcal{B}_G = \operatorname{Ob}\mathcal{B} \tag{4.3}$$

and

$$\mathcal{B}_G(x, y) = \mathcal{A}(Gx, Gy)$$

for all $x, y \in \operatorname{Ob}\mathcal{B}$. Composition law and identities in $\mathcal{B}$ are those in $\mathcal{A}$. G can be factored as $\mathcal{B} \xrightarrow{G^0} \mathcal{B}_G \xrightarrow{G^1} \mathcal{A}$; G^1 is full and faithful on morphisms and 2-cells.

Lemma 4.2.4. *Let $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$ be 2-functors which are part of a 2-adjunction $F \dashv G$. Let $\mathcal{B} \xrightarrow{G^0} \mathcal{B}_G \xrightarrow{G^1} \mathcal{A}$ with $\mathcal{B}_G$ as in (4.3). Suppose that*

- *(i) G is faithful on morphisms;*
- *(ii) for each parallel pair of morphisms $f, g : x \rightrightarrows y$ in $\mathcal{B}_G$ there is a morphism $p : [f \Rightarrow g] \to x$ in $\mathcal{B}$ such that $G^0(p)$ is part of a strong inserter from f to g; and*
- *(iii) with p is as in (ii) and $h : G^0 z \to [f \Rightarrow g]$ a morphism in $\mathcal{B}_G$, it holds that if $G^0(p) \circ h$ is in the image of G^0, then h is in the image of G^0.*

Then $\eta : \operatorname{Id} \to GF = G^1 G^0 F$ is the unit of a biadjunction $G^0 F \dashv G^1$.

Proof. Let $f : x \to G(y)$ be a morphism in $\mathcal{A}$ for some $x \in \operatorname{Ob}\mathcal{A}$ and $y \in \operatorname{Ob}\mathcal{B} = \operatorname{Ob}\mathcal{B}_G$. By the 2-universality of η_x, there is a unique morphism $\tilde{f} : F(x) \to y$ in $\mathcal{B}$ such that $G(\tilde{f}) \circ \eta_x = f$. It follows that with $\bar{f} = G^0(\tilde{f}) : G^0(F(x)) \to G^0(y)$ we have $G^1(\bar{f}) \circ \eta_x = f$ and hence that the functor

$$\phi : \mathcal{B}(G^0(F(x)), y) \to \mathcal{A}(x, G^1(y))$$

induced by η_x is surjective on objects.

Now let

$$\begin{tikzcd}
x \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] \arrow[d, "\eta_x"] & & G^1(y) \\
G(F(x)) \arrow[urr, shift left, "G^1(\bar{f})"] \arrow[urr, shift right, "G^1(\bar{g})"'] & &
\end{tikzcd}$$

be a diagram in $\mathcal{A}$ such that $f = \eta_x \circ G^1(\bar{f})$ and $g = \eta_x \circ G^1(\bar{g})$. By assumption (ii), there exists a morphism $p : [\bar{f} \Rightarrow \bar{g}] \to F(x)$ in $\mathcal{B}$ such that $G^0(p) : [\bar{f} \Rightarrow \bar{g}] \to F(x)$ is a strong inserter from $\bar{f}$ to $\bar{g}$ in $\mathcal{B}_G$. We then have a series of isomorphisms

$$\begin{aligned}
&\{\bar{\alpha} : \bar{f} \Rightarrow \bar{g}\} \\
\cong&\{[\bar{\alpha}] : F(x) \to [\bar{f} \Rightarrow \bar{g}] \text{ in } \mathcal{B}_G \mid G^0(p) \circ [\bar{\alpha}] = \mathrm{id}_{G(F(x))}\} \\
\cong&\{a : F(x) \to [\bar{f} \Rightarrow \bar{g}]) \mid p \circ a = \mathrm{id}_{F(x)}\} \\
\cong&\{[\alpha] : x \to G([\bar{f} \Rightarrow \bar{g}])) \mid p \circ [\alpha] = \eta_x\} \\
\cong&\{\alpha : \underbrace{G(\bar{f}) \circ \eta_x}_{=f} \Rightarrow \underbrace{G(\bar{g}) \circ \eta_x}_{=g}\}.
\end{aligned}$$

All isomorphisms are obtained either from the universal property of η_x or that of the inserter, except for the second one; here we used that if $G^0(p) \circ [\alpha] = \mathrm{id}_{G(F(x))}$, there exists (by (iii)) a unique (by (i)) morphism $a : F(x) \to [\bar{f} \Rightarrow \bar{g}]$ in $\mathcal{B}$ such that $[\bar{\alpha}] = G^0(a)$.

It follows that ϕ is also full and faithful and hence an equivalence of categories. □

4.3 Categories and linear sketches

We construct a theory whose models are precisely the categories. The 2-category of linear sketches LinSketch is defined and a 2-adjunction LinSketch $\leftrightarrows$ Cat is constructed.

Definition 4.3.1. The theories $\mathbb{T}_{\mathtt{LinSketch}}$ and $\mathbb{T}_{\mathrm{Cat}}$ of *linear sketches* and *categories* are given as follows. They share a signature $\Sigma_{\mathtt{LinSketch}} = \Sigma_{\mathrm{Cat}}$ with two sorts

$$\mathrm{Ob} \qquad\qquad \mathrm{Mor}$$

of *objects* and *morphisms* and operations

$$s : \mathrm{Mor} \to \mathrm{Ob} \qquad\qquad t : \mathrm{Mor} \to \mathrm{Ob}$$

$$\mathrm{id}_{-} : \mathrm{Ob} \to \mathrm{Mor} \qquad\qquad (- \circ -) : \mathrm{Mor} \times \mathrm{Mor} \to \mathrm{Mor}\,.$$

Here, the first two operations s and t assign morphisms their *source* and *target*, respectively.

For variables f of sort Mor and x, y of sort Ob, we introduce the abbreviation

$$f : x \to y \equiv s(f) = x \wedge t(f) = x.$$

Thus, $f : x \to y$ if and only if f is a morphism with source x and target y. The theory $\mathbb{T}_{\mathtt{LinSketch}}$ is given by the axioms

$$\frac{}{s(f)\downarrow} \qquad\qquad \frac{}{t(f)\downarrow}$$

$$\frac{i = \mathrm{id}_x}{i : x \to x} \qquad\qquad \frac{f : x \to y \qquad g : y' \to z \qquad h = g \circ f}{y = y' \qquad h : x \to z}\,.$$

$\mathbb{T}_{\mathrm{Cat}}$ is given by the axioms of $\mathbb{T}_{\mathtt{LinSketch}}$ and furthermore

$$\frac{}{\mathrm{id}_x\downarrow}$$

$$\frac{f' = f \circ \mathrm{id}_x}{f' = f} \qquad\qquad \frac{f' = \mathrm{id}_y \circ f}{f' = f}$$

$$\begin{array}{c} t(f) = s(g) \\ \hline\hline (g \circ f)\downarrow \end{array} \qquad\qquad \frac{t(f) = s(g) \qquad t(g) = s(h)}{h \circ (g \circ f) = (h \circ g) \circ f}$$

Clearly we have $\mathrm{Cat}_\circ = \mathrm{Mod}(\mathbb{T}_{\mathrm{Cat}})$, where $\mathrm{Cat}_\circ$ is the 1-category underlying the 2-category Cat of small categories, functors and natural transformations. We define a 1-category $\mathrm{LinSketch}_\circ = \mathrm{Mod}(\mathbb{T}_{\mathtt{LinSketch}})$ in anticipation of the 2-category LinSketch (definition 4.3.5). From the inclusion $\mathbb{T}_{\mathtt{LinSketch}} \subseteq \mathbb{T}_{\mathrm{Cat}}$ we obtain an adjunction $\mathrm{LinSketch}_\circ \leftrightarrows \mathrm{Cat}_\circ$ that exhibts $\mathrm{Cat}_\circ$ as full reflective subcategory of $\mathrm{LinSketch}_\circ$.

Let $F, G : \mathcal{S} \rightrightarrows \mathcal{C}$ be a parallel pair of morphisms in $\mathrm{LinSketch}_\circ$ with $\mathcal{C}$ a category. A *natural transformation* $\alpha : F \Rightarrow G$ from F to G consists of a family of morphisms $\alpha_x : F(x) \to G(x)$ in $\mathcal{C}$ for $x \in \mathrm{Ob}\,\mathcal{S}$ such that for each morphism $f : x \to y$ in $\mathcal{S}$ (note that every morphism $f \in \mathrm{Mor}\,\mathcal{S}$ can be assigned a signature!)

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "\alpha_x"] & F(y) \arrow[d, "\alpha_y"] \\ G(x) \arrow[r, "G(f)"] & G(y) \end{tikzcd} \tag{4.4}$$

is a commuting square in $\mathcal{C}$. It is easy to verify that with the composition law

$$(\beta\alpha)_x = \beta_x \alpha_x$$

we obtain a category $\mathrm{LinSketch}(\mathcal{S}, \mathcal{C})$ of $\mathrm{LinSketch}_\circ$ morphisms and their natural transformations.

Definition 4.3.2. The 2-category $\mathrm{LinSketch}^-$ is given by

$$\mathrm{Ob}\,\mathrm{LinSketch}^- = \mathrm{Ob}\,\mathrm{LinSketch}_\circ$$

and

$$\mathrm{LinSketch}^-(\mathcal{S}, \mathcal{T}) = \begin{cases} \mathrm{LinSketch}(\mathcal{S}, \mathcal{T}) & \text{if } \mathcal{T} \in \mathrm{Ob}\,\mathrm{Cat}_\circ \\ \emptyset & \text{otherwise.} \end{cases}$$

Composition of morphisms is inherited from LinSketch. Horizontal composition of 2-cells

$$\begin{tikzcd}
\mathcal{S} \arrow[rr, "F_1"] & {} \arrow[d, Rightarrow, "\alpha"] & \mathcal{C} \arrow[rr, "G_2"] & {} \arrow[d, Rightarrow, "\beta"] & \mathcal{D} \\
\mathcal{S} \arrow[rr, "F_2"'] & {} & \mathcal{C} \arrow[rr, "G_2"'] & {} & \mathcal{D}
\end{tikzcd}$$

is given by

$$(\beta \circ \alpha)_x = \beta_{F_2(x)} \circ G_1(\alpha_x) = G_2(\alpha_x) \circ \beta_{F_1(x)}.$$

We have an evident 2-functor Cat → $\mathrm{LinSketch}^-$ which is a full and locally full inclusion of 2-categories, i.e. it is injective on objects and full and faithful on morphisms and 2-cells.

Proposition 4.3.3. Cat *admits strong inserters to every pair of parallel pair of functors* $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ *in* Cat, *which are preserved by the forgetful 2-functor* Cat → $\mathrm{LinSketch}^-$. *The same holds true for strong iso-inserters.*

Proof. We will only verify the statement about strong inserters, the proof for the strong iso-inserters being analogous. The objects of our strong inserter candidate $[F \Rightarrow G]$ are given by

$$\mathrm{Ob}[F \Rightarrow G] = \{(x, d) \in \mathrm{Ob}\,\mathcal{C} \times \mathrm{Mor}\,\mathcal{D} \mid d : F(x) \to G(y)\}.$$

(For the strong iso-inserter, we demand that d is an isomorphism.) A morphism $(x, d) \to (y, e)$ in $[F \Rightarrow G]$ is a morphism $f : x \to y$ in $\mathcal{C}$ such that

$$\begin{tikzcd}
F(x) \arrow[r, "F(f)"] \arrow[d, "d"] & F(y) \arrow[d, "e"] \\
G(x) \arrow[r, "G(f)"] & G(y)
\end{tikzcd}$$

commutes in $\mathcal{D}$. Composition and identities are inherited from $\mathcal{C}$. We have an evident projection $P : [F \Rightarrow G] \to \mathcal{C}$ given by the first component of objects (x, d) and a natural transformation $\alpha : FP \Rightarrow GP$ given by the second component $\alpha_{(x,d)} = d$.

Note that, while P is faithful, it is not injective on objects. Thus, the signature $f : (x, d) \to (y, e)$ of a morphism in $[F \Rightarrow G]$ cannot generally be recovered from its image $Pf = f : x \to y$ in $\mathcal{C}$, although our notation might suggest that this is possible.

Now let $\mathcal{S}$ be linear sketch, let $Q : \mathcal{S} \to \mathcal{C}$ a morphism of linear sketches and let $\beta : FQ \Rightarrow GQ$ a natural transformation. (P, α) is a strong inserter and preserved by the 2-inclusion Cat →

LinSketch$^-$ if we can verify that there exists a unique morphism $[\beta] : \mathcal{S} \to \mathcal{C}$ such that $Q = P[\beta]$ and $\beta = \alpha[\beta]$.

Suppose for the moment that $[\beta]$ is already defined and has the required properties. From $Q = P[\beta]$ it follows that for every object $x \in \mathcal{S}$, we have $[\beta](x) = (Q(x), d)$ for some $d : F(Q(x)) \to G(Q(x))$. Then from

$$d = \alpha_{(Q(x),d)} = \alpha_{([\beta]x)} = (\alpha[\beta])_x = \beta_x$$

we conclude more precisely that $[\beta](x) = (Q(x), \beta_x)$. Every morphism $f \in \operatorname{Mor} \mathcal{S}$ can be assigned a signature $f : x \to y$ and by faithfulness of P it follows that $[\beta](f) = Q(f) : (x, \beta_x) \to (y, \beta_y)$.

Indeed, the assignments $[\beta](x) = (Q(x), \beta_x)$ and $[\beta](f) = Q(f) : (x, \beta_x) \to (y, \beta_y)$ constitute a well-defined $\{\mathrm{Ob}, \mathrm{Mor}\}$-sorted map by the naturality square (4.4) for β. This map satisfies $Q = P[\beta]$ and $\beta = \alpha[\beta]$ and would by the above reasoning be the unique morphism with these properties. All that remains is the verification that $[\beta]$ respects the four operations of linear sketches and categories.

For source and target, this is by definition. If $i = \mathrm{id}_x$ in $\mathcal{S}$, then from $i : x \to x$ it follows that $[\beta](i) : [\beta](x) \to [\beta](x)$ and hence that $[\beta](i)$ is the identity at $[\beta](x)$. Similarly, every triple f, g, h in $\mathcal{S}$ such that $h = g \circ f$ can be assigned a signature

$$\begin{tikzcd}
x \arrow[r, "f"'] \arrow[rr, bend left, "h"] & y \arrow[r, "g"'] & z,
\end{tikzcd}$$

hence $[\beta](f) : [\beta](x) \to [\beta](y)$ and $[\beta](g) : [\beta](y) \to [\beta](z)$ which are thus composable. Similarly, $[\beta](h) : [\beta](x) \to [\beta](z)$ whence

$$Q(h) = [\beta](h) = [\beta](g) \circ [\beta](f) = Q(g) \circ Q(f).$$

☐

Corollary 4.3.4. *The forgetful 2-functor* Cat $\to$ LinSketch$^-$ *is a right 2-adjoint. The 1-functor underlying its left 2-adjoint is the restriction of the free category functor* LinSketch$_\circ$ $\to$ Cat$_\circ$.

Proof. Combine 4.2.3 and 4.3.3. ☐

We are now ready to define the promised 2-category LinSketch whose underlying 1-category is the already defined category LinSketch$_\circ$.

Definition 4.3.5. The 2-category LinSketch is given on objects and morphisms by LinSketch. A 2-cell

$$\begin{tikzcd}
\mathcal{S} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{T}
\end{tikzcd}$$

is a 2-cell

$$\begin{tikzcd}
\mathcal{S} \arrow[r, bend left=50, "\eta_{\mathcal{T}} F"] \arrow[r, bend right=50, "\eta_{\mathcal{T}} G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{T}'
\end{tikzcd}$$

in LinSketch$^-$, where $\eta_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}'$ is the unit of the 2-adjunction LinSketch$^-$ $\leftrightarrows$ Cat.

Let

$$\begin{tikzcd}
\mathcal{S} \arrow[r, bend left=50, "F_\circ"] \arrow[r, bend right=50, "F_2"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{T} \arrow[r, bend left=50, "G_\circ"] \arrow[r, bend right=50, "G_2"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{U}
\end{tikzcd}$$

be a pair of horizontally composable 2-cells in LinSketch. By 4.3.4, we can lift the 2-cell $\beta : \eta_{\mathcal{U}} G_\circ \Rightarrow \eta_{\mathcal{U}} U_2$ in LinSketch$^-$ uniquely to a 2-cell β' such that

$$\begin{tikzcd}
\mathcal{T} \arrow[r, "\eta_{\mathcal{T}}"] & \mathcal{T}' \arrow[r, bend left, "G'_\circ"] \arrow[r, bend right, "G'_2"'] \arrow[r, phantom, "\Downarrow\beta'"] & \mathcal{U} & = & \mathcal{T} \arrow[r, bend left, "G_\circ"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{U}
\end{tikzcd}$$

Then composition $\beta \circ \alpha$ in LinSketch is given composition $\beta' \circ \alpha$ in LinSketch$^-$.

Now the same argument that proved 4.3.4 also proves the following corollary.

Corollary 4.3.6. *The forgetful 2-functor* Cat $\to$ LinSketch *is a right 2-adjoint. The functor underlying its left 2-adjoint is the free category functor* LinSketch$_\circ$ $\to$ Cat$_\circ$. □

4.4 Left exact categories and left exact sketches

Left exact categories are finitely complete categories, or, equivalently, categories with initial object such that all cospans can be completed to pullback squares. Another commonly used term is *cartesian* categories [12]. As usual when working without additional choice principles, in particular when working constructively, we do not require that terminal objects and pullbacks merely *exist* but that left exact categories come with *assigned* terminal objects and pullbacks, i.e. a chosen initial object and a function assigning to each cospan a completion to a pullback square is part of the data. This additional datum will be modelled by operations in the corresponding theory, which means that morphisms of models will have to preserve the assigned limits. Left exact functors, however, are usually required to preserve these only up to isomorphism. Although this means that certain left adjoints cannot exist, we use the machinery of 4.2 to construct various 2-adjunctions and biadjunctions; see corollary 4.4.4.

Definition 4.4.1. We define theories

(i) $\mathbb{T}_{\mathrm{LexSketch}} = (\mathbb{T}_{\mathrm{LexSketch}}, \Sigma_{\mathrm{LexSketch}})$ of *left exact sketches*,

(ii) $\mathbb{T}_{\mathrm{sLexSketch}} = (\mathbb{T}_{\mathrm{sLexSketch}}, \Sigma_{\mathrm{sLexSketch}})$ of *strict left exact sketches* and

(iii) $\mathbb{T}_{\mathrm{sLex}} = (\mathbb{T}_{\mathrm{sLex}}, \Sigma_{\mathrm{sLexSketch}})$ of *strict left exact categories*

fitting into a commutative square of theory inclusions

$$\begin{tikzcd}
\mathbb{T}_{\mathrm{LinSketch}} \arrow[rr, hook] \arrow[d, hook] & & \mathbb{T}_{\mathrm{Cat}} \arrow[d, hook] \\
\mathbb{T}_{\mathrm{LexSketch}} \arrow[r, hook] & \mathbb{T}_{\mathrm{sLexSketch}} \arrow[r, hook] & \mathbb{T}_{\mathrm{sLex}}
\end{tikzcd}$$

(i). $\Sigma_{\mathrm{LexSketch}}$ extends $\Sigma_{\mathrm{LinSketch}}$ by sorts

$$\mathrm{Term} \qquad\qquad \mathrm{Pb}$$

of *terminal objects* and *pullback squares* and operations

$$o : \mathrm{Term} \to \mathrm{Ob}$$

$$\mathrm{pr}_1 : \mathrm{Pb} \to \mathrm{Mor} \qquad \mathrm{pr}_2 : \mathrm{Pb} \to \mathrm{Mor} \qquad \ell_1 : \mathrm{Pb} \to \mathrm{Mor}\,. \qquad \ell_2 : \mathrm{Pb} \to \mathrm{Mor}\,.$$

which we think of as assigning to each terminal object $x \in \mathrm{Term}\,\mathcal{C}$ its underlying object and to each pullback squares $p \in \mathrm{Pb}\,\mathcal{C}$ the underlying commutative square

$$\begin{tikzcd} \cdot \arrow[r, "\mathrm{pr}_2(p)"] \arrow[d, "\mathrm{pr}_1(p)"'] \arrow[dr, phantom, "p"] & \cdot \arrow[d, "\ell_2(p)"] \\ \cdot \arrow[r, "\ell_1(p)"'] & \cdot \end{tikzcd} \tag{4.5}$$

The axioms of $\mathbb{T}_{\mathtt{LexSketch}}$ are given by those of $\mathbb{T}_{\mathtt{LinSketch}}$ and furthermore

$$\frac{}{s(\mathrm{pr}_1(p)) = s(\mathrm{pr}_2(p))} \qquad \frac{}{t(\mathrm{pr}_1(p)) = s(\ell_1(p))}$$

$$\frac{}{t(\mathrm{pr}_2(p)) = s(\ell_2(p))} \qquad \frac{}{t(\ell_1(p)) = t(\ell_2(p))}$$

In particular, all new operations are total and the $\mathrm{pr}_i(p), \ell_i(p)$ can be assembled into squares such as (4.5), although they do not need to be commutative; the two possible composites need not even exist.

(ii). $\Sigma_{\mathtt{sLexSketch}}$ extends $\Sigma_{\mathtt{LexSketch}}$ by operations

$$\top : \mathrm{Term} \qquad (- \times -) : \mathrm{Mor} \times \mathrm{Mor} \to \mathrm{Pb}$$

which we think of as a canonical terminal object and canonical completions of cospans to pullback squares. In addition to the axioms of $\mathbb{T}_{\mathtt{LexSketch}}$, we enforce the axiom

$$\frac{p = f_1 \times f_2}{\ell_1(p) = f_1 \qquad \ell_2(p) = f_2}$$

so that the canonical pullback square over a given cospan is not only some pullback square but also completes the given cospan.

(iii). $\Sigma_{\mathtt{sLex}}$ extends $\Sigma_{\mathtt{LexSketch}}$ by operations

$$!_{-}(-) : \mathrm{Term} \times \mathrm{Ob} \to \mathrm{Mor} \qquad \langle -, - \rangle_{-} : \mathrm{Mor} \times \mathrm{Mor} \times \mathrm{Pb} \to \mathrm{Mor}$$

$$\mathrm{term} : \mathrm{Term} \times \mathrm{Mor} \times \mathrm{Mor} \to \mathrm{Term} \qquad \mathrm{pb} : \mathrm{Pb} \times \mathrm{Mor} \times \mathrm{Mor} \to \mathrm{Pb}\,.$$

Their purpose is as follows. The universal properties of terminal objects and pullback squares assert the unique existence of certain morphisms, which will be given by the first two operations. The latter two operations are there to make sure that Term and Pb are closed under isomorphism so that for example if $f : y \rightleftarrows o(x') : g$ is an isomorphism in a model $\mathcal{C}$, then $o(\mathrm{term}(x', f, g)) = y$. To this end, we introduce the abbreviation

$$\mathrm{Iso}\, f\, g = g \circ f = \mathrm{id}_{s(f))} \wedge f \circ g = \mathrm{id}_{s(g))}\,.$$

In addition to the axioms of $\mathbb{T}_{\mathtt{Cat}}$ and $\mathbb{T}_{\mathtt{sLexSketch}}$, we add axioms

$$\frac{}{o(x)\downarrow} \qquad \frac{o(x) = o(y)}{x = y}$$

$$\frac{}{!_x(y)\downarrow} \qquad \frac{}{!_x(y) : y \to o(x)} \qquad \frac{f : y \to o(x)}{f = !_x(y)}$$

$$\frac{}{\top\downarrow}$$

$$\begin{array}{c} \mathrm{Iso}\, f\ g \qquad t(f) = o(x) \\ \hline\hline \mathrm{term}(x, f, g)\downarrow \end{array} \qquad\qquad \frac{x' = \mathrm{term}(x, f, g)}{o(x') = s(f)}$$

governing terminal objects and

$$\frac{}{\ell_1(p) \circ \mathrm{pr}_1(p) = \ell_2(p) \circ \mathrm{pr}_2(p)}$$

$$\frac{\mathrm{pr}_1(p) = \mathrm{pr}_1(p') \qquad \mathrm{pr}_2(p) = \mathrm{pr}_2(p') \qquad \ell_1(p) = \ell_1(p') \qquad \ell_2(p) = \ell_2(p')}{p = p'}$$

$$\begin{array}{c} q_1 \circ \ell_1(p) = q_2 \circ \ell_2(p) \\ \hline\hline \langle q_1, q_2\rangle_p \downarrow \end{array} \qquad\qquad \frac{k = \langle q_1, q_2\rangle_p}{\mathrm{pr}_1(p) \circ k = q_1 \qquad \mathrm{pr}_2(p) \circ k = q_2}$$

$$\frac{q_1 = \mathrm{pr}_1(p) \circ k \qquad q_2 = \mathrm{pr}_2(p) \circ k}{k = \langle q_1, q_2\rangle_p}$$

$$\begin{array}{c} t(f_1) = t(f_2) \\ \hline\hline (f_1 \times f_2)\downarrow \end{array} \qquad\qquad \frac{p = (f_1 \times f_2)}{\ell_1(p) = f_1 \qquad \ell_2(p) = f_2}$$

$$\begin{array}{c} \mathrm{Iso}\, f\ g \qquad t(f) = s(\mathrm{pr}_1(p)) \\ \hline\hline \mathrm{pb}(p, f, g)\downarrow \end{array}$$

$$\frac{p' = \mathrm{pb}(p, f, g)}{\mathrm{pr}_1(p') = \mathrm{pr}_1(p) \circ f \qquad \mathrm{pr}_2(p') = \mathrm{pr}_2(p) \circ f \qquad \ell_1(p') = \ell_1(p) \qquad \ell_2(p') = \ell_2(p)}$$

governing pullback squares.

We abbreviate

$$\mathrm{LexSketch}_\circ = \mathrm{Mod}(\mathbb{T}_{\mathrm{LexSketch}}), \qquad\qquad \mathrm{sLexSketch}_\circ = \mathrm{Mod}(\mathbb{T}_{\mathrm{sLexSketch}})$$

and

$$\mathrm{sLex}_\circ = \mathrm{Mod}(\mathbb{T}_{\mathrm{sLex}}).$$

The objects of sLex will be called *left exact categories* and morphism in sLex are *strict left exact functors.* Note that left exact categories are what might also be called *strict* left exact categories; thus our left exact categories always come with canonical choices of terminal object pullback squares given by $\top$ and $(- \times -)$, which strict left exact functors have to preserve.

These three 1-categories all come with forgetful functors to $\mathrm{LinSketch}_\circ$, and we extend them to 2-functors by *definining* them to be full and faithful on 2-cells. Thus, for example in sLex, a 2-cell $\alpha : F \Rightarrow G$ is by definition a 2-cell in LinSketch, i.e. a natural transformation of underlying functors.

We now have a forgetful 2-functor $\mathcal{V} : \mathrm{sLex} \to \mathrm{LexSketch}$, which we use in the definition of the category of *left exact categories and left exact functors*

$$\mathrm{Lex} := \mathrm{sLex}_{\mathcal{V}},$$

see (4.3). Objects of Lex are left exact categories with canonical choices for terminal object and pullback squares while left exact functors have to preserve finite limits but not necessarily their canonical choices.

The forgetful 2-functors in the square

$$\begin{tikzcd}\mathrm{sLex} \arrow[r] \arrow[d] & \mathrm{sLexSketch} \arrow[d] \\ \mathrm{Lex} \arrow[r] & \mathrm{LexSketch}\end{tikzcd} \tag{4.6}$$

are all faithful on morphisms and full and faithful on 2-cells. We thus identify all morphisms and 2-cells in these categories with their image in LexSketch and and say, for example, that a morphisms $\mathcal{S} \to \mathcal{C}$ in LexSketch with $\mathcal{S}$ a strict left exact sketch and $\mathcal{C}$ a left exact category is strict left exact if it is in the image of the forgetful functor sLexSketch $\to$ LexSketch.

The lower horizontal forgetful functor $\mathrm{Lex}_\circ \to \mathrm{LexSketch}_\circ$ from (4.6) is by definition full and faithful. The same is true for the upper forgetful functor $\mathrm{sLex}_\circ \to \mathrm{sLexSketch}_\circ$, but this requires proof:

Lemma 4.4.2. *The forgetful functor* $\mathrm{sLex}_\circ \to \mathrm{sLexSketch}_\circ$ *is full and faithful.*

Proof. It was observed earlier that the functor is faithful. Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of strict left exact sketches with $\mathcal{C}$ and $\mathcal{D}$ arising from left exact categories. We will prove that then F preserves $\langle -, - \rangle_-$ and pb; the verification for $!_-$ and term is analogous and in fact easier.

Suppose $k = \langle q_1, q_2 \rangle_p \in \mathcal{C}$. Then

$$F(\ell_i(p) \circ q_i) = \ell_i(F(p)) \circ F(q_i)$$

for $i \in \{1, 2\}$, and so $k' = \langle F(q_1), F(q_2) \rangle_{F(p)}$ is defined. Then from

$$\mathrm{pr}_i(F(p)) \circ F(k) = F(\mathrm{pr}_i(p) \circ k) = F(q_i)$$

for $i \in \{1, 2\}$ we can conclude that $F(k) = k'$.

Now let $p' = \mathrm{pb}(p, f, g) \in \mathrm{Pb}\,\mathcal{C}$. Isomorphisms are preserved by functors, thus

$$\mathrm{pb}(F(p), F(f), F(g)) \downarrow .$$

We have

$$\mathrm{pr}_i(F(p')) = F(\mathrm{pr}_i(p')) = F(\mathrm{pr}_i(p) \circ f) = \mathrm{pr}_i(F(p)) \circ F(f)$$

for $i \in \{1, 2\}$ and it follows by the joint injectivity of the pr_i and ℓ_i that

$$F(p') = \mathrm{pb}(F(p), F(f), F(g)).$$

□

If $\mathcal{C}$ is a left exact category and $\mathcal{S}$ a left exact sketch, there is at most one morphism of left exact sketches extending a morphism of linear sketches $F : \mathcal{S} \to \mathcal{C}$. For, in $\mathcal{C}$, terminal objects in are equal if their underlying objects are equal, and pullback squares are equal if their underlying commutative square is equal. Thus, if $\mathrm{Ob}\,F$ and $\mathrm{Mor}\,F$ are fixed, we have at most one choice for $\mathrm{Term}\,F$ and $\mathrm{Pb}\,F$ if F is to preserve o and the pr_i, ℓ_i.

Consequently, we identify morphisms in LexSketch with codomain arising from a strict left exact category with their image in LinSketch and say that a morphism of linear sketches is “left exact” or “strict left exact” if it has a (necessarily unique) preimage under the maps $\mathrm{LexSketch}_\circ(\mathcal{S}, \mathcal{C}) \hookrightarrow \mathrm{LinSketch}_\circ(\mathcal{S}, \mathcal{C})$ or even $\mathrm{sLexSketch}_\circ(\mathcal{S}, \mathcal{C}) \hookrightarrow \mathrm{LinSketch}_\circ(\mathcal{S}, \mathcal{C})$.

Proposition 4.4.3. *Let $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ be a pair of parallel left exact functors.*

(i) The strong inserter $[F \Rightarrow G]$ in Cat *can be endowed with the structure of a left exact category in such a way that the projection $P : [F \Rightarrow G] \to \mathcal{C}$ is strict left exact.*

(ii) Let $\mathcal{S}$ be a left exact sketch, and let $H : \mathcal{S} \to [F \Rightarrow G]$ be a morphism of linear sketches. Then

$$H \text{ is left exact} \iff PH \text{ is left exact.}$$

If $\mathcal{S}$ is a strict left exact sketch, then

$$H \text{ is strict left exact} \iff PH \text{ is strict left exact.}$$

(iii) Let $\alpha : PF \Rightarrow PF$ be the canonical natural transformation. The pair (P, α) is a strong inserter from F to G in Lex. *Its universal property is preserved by the forgetful 2-functor* Lex → LexSketch.

If F and G are strict left exact, then (P, α) is a strong inserter in sLex. *Its universal property is preserved by the forgetful 2-functors* sLex → sLexSketch *and* sLex → LexSketch.

Analogous statements hold for strong iso-inserters.

Proof. We will only prove the statements about strong inserters, the case of strong iso-inserters is analogous.

(i). The terminal objects of $[F \Rightarrow G]$ are given by tuples (t, u), where t is terminal in $\mathcal{C}$ and $u : F(t) \to G(t)$ it the unique morphism from $F(t)$ to the terminal object $G(t)$. The canonical terminal object in $[F \Rightarrow G]$ is given by (t, u) as above with t the canonical terminal object in $\mathcal{C}$.

We verify the universal property of a terminal object (t, u) in $[F \Rightarrow G]$. Let $(x, d) \in \mathrm{Ob}[F \Rightarrow G]$ be an arbitrary object. Then there is a unique morphism $! : x \to t$ in $\mathcal{C}$ and

$$\begin{tikzcd}
F(x) \arrow[r, "F(!)"] \arrow[d, "d"] & F(t) \arrow[d, "u"] \\
G(x) \arrow[r, "G(!)"] & G(t)
\end{tikzcd}$$

commutes by the universal property of the terminal object $G(t)$.

Pullback squares in $[F \Rightarrow G]$ are squares

$$\begin{tikzcd}
(x_1 \times_y x_2, u) \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "p"] & (x_2, d_2) \arrow[d, "f_2"] \\
(x_1, d_1) \arrow[r, "f_1"'] & (y, e)
\end{tikzcd} \tag{4.7}$$

which are mapped to pullback squares p by P; they are then automatically commutative in $[F \Rightarrow G]$. Note that the morphism $u : F(x_1 \times_y x_2) \to G(x_1 \times_y x_2)$ in $\mathcal{D}$ of such a square is uniquely determined by the rest of the data. For, from

$$G(p_1) \circ u = d_1 \circ F(p_1) \qquad G(p_2) \circ u = d_2 \circ F(p_2)$$

and

$$\begin{aligned} G(f_1) \circ d_1 \circ F(p_1) &= e \circ F(f_1 \circ p_1) \\ &= e \circ F(f_2 \circ p_2) \\ &= G(f_2) \circ d_1 \circ F(p_1) \end{aligned}$$

it follows that

$$u = \langle d_1 \circ F(p_1), d_2 \circ F(p_2)\rangle_{G(p)}.$$

Taking the last equation as definition, we obtain the canonical pullback squares in $[F \Rightarrow G]$ which are uniquely determined by the requirement that P preserves canonical pullback squares.

Let us prove the universal property of the commutative squares (4.7). Let

$$\begin{tikzcd}
(z, v) \arrow[r, "q_2"] \arrow[d, "q_1"] & (x_2, d_2) \arrow[d, "f_2"] \\
(x_1, d_1) \arrow[r, "f_1"] & (y, e)
\end{tikzcd}$$

be another commutative square in $[F \Rightarrow G]$. If $k : (z, v) \to (x_1 \times_y x_2, u)$ is compatible with the q_i and p_i in $\mathcal{D}$, then in particular $k : z \to x_1 \times_y x_2$ is compatible with them in $\mathcal{C}$. Thus, we have no choice but verify that $k = \langle q_1, q_2\rangle_p : (z, v) \to (x_1 \times_y x_2, u)$ as morphism in $[F \Rightarrow G]$.

In $\mathcal{D}$,

$$\begin{aligned}
&\mathrm{pr}_i(G(p)) \circ G(k) \circ v \\
=&G(q_i) \circ v \\
=&d_i \circ F(q_i) \\
=&d_i \circ F(\mathrm{pr}_i(p)) \circ F(k) \\
=&\mathrm{pr}_i(G(p)) \circ u \circ F(k)
\end{aligned}$$

for $i \in \{1, 2\}$ and hence

$$G(k) \circ v = u \circ F(k) = \langle G(q_1) \circ v, G(q_2) \circ v\rangle_{G(p)}$$

by the universal property of $G(p)$. Thus $k : (z, v) \to (x_1 \times_y x_2, u)$ is well-defined.

(ii). The directions from left to right are trivial. Thus, suppose that PH is (not necessarily strict) left exact. We have to show that then already H is left exact. Let $t \in \mathrm{Ob}\,\mathcal{S}$ be a terminal object. Then $H(t) = (P(H(t)), u)$ for some $u : F(P(H(t))) \to G(P(H(t)))$. But $P(H(t))$ and hence $G(P(H(t)))$ are terminal, so there is only one such u. It follows from the proof of *(i)* that then $H(t) = (P(H(t)), u)$ is terminal.

Every pullback square p in $\mathcal{S}$ has an underlying square

$$\begin{tikzcd}
x_1 \times_y x_2 \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "p"] & x_2 \arrow[d, "f_2"] \\
x_1 \arrow[r, "f_1"'] & y
\end{tikzcd} \tag{4.8}$$

which is thus mapped to a square

$$\begin{tikzcd}
H(x_1 \times_y x_2) \arrow[r, "H(p_2)"] \arrow[d, "H(p_1)"'] & H(x_2) \arrow[d, "H(f_2)"] \\
H(x_1) \arrow[r, "H(f_1)"'] & H(y)
\end{tikzcd} \tag{4.9}$$

in $[F \Rightarrow G]$ by H. By assumption, (4.9) is mapped to a pullback square in $\mathcal{C}$, and is hence itself a pullback square.

Now suppose that PH is *strict* left exact. We have already proved that then H is left exact. Given lemma 4.4.2, it suffices to prove that H preserves canonical terminal objects and pullback squares. There is only one object $(t, u) \in \mathrm{Ob}[F \Rightarrow G]$ such that t is the canonical terminal object in $\mathcal{C}$ and so H preserves the terminal object of $\mathcal{S}$ if it exists. If $p = f_1 \times f_2$ is a canonical pullback square in $\mathcal{S}$, then p can be assigned an underlying commutative square (4.8) because $\ell_i(f_1 \times f_2) = f_i$ for $i \in \{1, 2\}$. The pullback square (4.9) is the canonical pullback square over $H(f_1), H(f_2)$ because it is mapped to the canonical pullback square over $P(H(f_1)), P(H(f_2))$.

(iii). Let $\mathcal{S}$ be a left exact sketch, $Q : \mathcal{S} \to \mathcal{C}$ be a morphism of left exact sketches and $\beta : FQ \Rightarrow GQ$ be a natural transformation. Then the induced morphism $[\beta] : \mathcal{S} \to [F \Rightarrow G]$ of linear sketches is left exact by *(ii)* because $P[\beta] = Q$ is left exact. Thus, (P, α) is a strong inserter from F to G in LexSketch and hence also in Lex.

Now suppose F and G are strict left exact, $\mathcal{S}$ is a strict left exact sketch and Q strict left exact. Then the induced morphism $[\beta]$ is strict left exact, again by *(ii)*. Thus, (P, α) is a strong inserter in sLexSketch and hence in sLex because the latter is a full subcategory of the former.

The preservation of strong inserters by the forgetful 2-functor sLex → LexSketch follows from the same preservation property of Lex → LexSketch. □

Corollary 4.4.4. *The 2-functors in the diagram*

$$\begin{tikzcd}
\mathrm{sLex} \arrow[r, "\diamond"] \arrow[rr, bend left, "\diamond"] \arrow[d, "\diamond"] & \mathrm{Lex} \arrow[r] \arrow[d] & \mathrm{Cat} \arrow[d, "\diamond"] \\
\mathrm{sLexSketch} \arrow[r, "\diamond"] & \mathrm{LexSketch} \arrow[r, "\diamond"] & \mathrm{LinSketch}
\end{tikzcd}$$

are right biadjoints. The 2-functors marked with ◇ are right 2-adjoints.

Proof. For Cat → LinSketch, this is exactly 4.3.6.

The units of the 1-categorical adjunctions $\mathrm{LexSketch}_\circ \leftrightarrows \mathrm{LinSketch}_\circ$ and $\mathrm{sLexSketch}_\circ \leftrightarrows \mathrm{LexSketch}_\circ$ are isomorphisms and the corresponding forgetful 2-functors full and faithful on 2-cells by definition, thus 4.2.1 applies.

The forgetful 2-functors sLex → sLexSketch preserves strong inserters by 4.4.3. Thus, its left 2-adjoint can be obtained by an application of 4.2.3.

Because adjunctions compose (or again by an application of 4.2.3), the 2-functor $G : \mathrm{sLex} \to \mathrm{LexSketch}$ is right 2-adjoint. We have $\mathrm{Lex} = \mathrm{sLex}_G$ and the corresponding factorization $\mathrm{sLex} \xrightarrow{G^0} \mathrm{Lex} \xrightarrow{G^1} \mathrm{LexSketch}$ satisfies the premises of 4.2.4, by which we conclude that G^1 is a right biadjoint.

The composite 2-functors

$$\mathrm{sLex} \to \mathrm{Lex} \to \mathrm{LexSketch}$$

and

$$\mathrm{sLex} \to \mathrm{Cat} \to \mathrm{LinSketch}$$

are right 2-adjoints while the composite

$$\mathrm{Lex} \to \mathrm{Cat} \to \mathrm{LinSketch}$$

is a right biadjoint. In each case, the second 2-functor is fully faithful on morphisms and 2-cells. By 4.2.2, sLex → Lex and sLex → Cat are right 2-adjoints while Lex → Cat is a right biadjoint. □

4.5 Locally cartesian closed categories and locally cartesian closed sketches

Binary products in slice categories $\mathcal{C}_{/s}$, i.e. products “local” over s, are given by pullback squares over cospans $\cdot \to s \leftarrow \cdot$ in $\mathcal{C}$. Thus, the slice categories $\mathcal{C}_{/s}$ admit finite products for all s if and only if $\mathcal{C}$ admits pullback squares to all cospans. *Locally cartesian closed (lcc) categories* are left exact categories such that the slice categories $\mathcal{C}_{/s}$ are cartesian *closed*. For lack of an established term, we will refer to the internal homs in slice categories as *local* internal homs.

Note that the definition implies that lcc categories also have terminal objects, although this is not reflected in the name. Thus, locally cartesian closed categories are, in particular, cartesian closed because if s is terminal in $\mathcal{C}$, then $\mathcal{C}_{/s} \cong \mathcal{C}$.

The relevance of lcc categories is due to them being natural models of ML type theory supporting Π and Σ types. Using the correspondence of ML type theory and lcc categories, it follows from the undecidability of terms in ML type theory over a single base type that the corresponding bifree lcc category is undecidable [6]. Translating the type theoretic proof into categorical language, we present a direct proof.

Definition 4.5.1. We define theories

(i) $\mathbb{T}_{\mathrm{LccSketch}} = (\Sigma_{\mathrm{LccSketch}}, \mathbb{T}_{\mathrm{LccSketch}})$ of *locally cartesian closed sketches*,

(ii) $\mathbb{T}_{\mathrm{sLccSketch}} = (\Sigma_{\mathrm{sLccSketch}}, \mathbb{T}_{\mathrm{sLccSketch}})$ of *strict locally cartesian closed sketches* and

(iii) $\mathbb{T}_{\mathrm{sLcc}} = (\Sigma_{\mathrm{sLcc}}, \mathbb{T}_{\mathrm{sLcc}})$ of *strict locally cartesian closed categories*

fitting into a commutative diagram of theory extensions

$$\begin{tikzcd}
\mathbb{T}_{\mathrm{LexSketch}} \arrow[r, hook] \arrow[d, hook] & \mathbb{T}_{\mathrm{sLexSketch}} \arrow[r, hook] \arrow[d, hook] & \mathbb{T}_{\mathrm{sLex}} \arrow[d, hook] \\
\mathbb{T}_{\mathrm{LccSketch}} \arrow[r, hook] & \mathbb{T}_{\mathrm{sLccSketch}} \arrow[r, hook] & \mathbb{T}_{\mathrm{sLcc}}.
\end{tikzcd}$$

(i). The signature $\Sigma_{\mathrm{LccSketch}}$ extends $\Sigma_{\mathrm{LexSketch}}$ by a single sort

$$\mathrm{Hom},$$

which we call the sort of *local internal homs*, or just internal homs. Let $f : x \to s \leftarrow y : g$ be a cospan. An internal hom h from f to g comes with an underlying morphism $[h] : z \to s$ with codomain s and evaluation maps $\varepsilon(h, p)$ fitting into diagrams

$$\begin{tikzcd}
& x \times_s z \arrow[ld] \arrow[rd] \arrow[rr, "\varepsilon(h,p)"] & & y \arrow[lldd, bend left, "g"] \\
x \arrow[rd, "f"] & p & z \arrow[ld, "{[h]}"'] & \\
& s & &
\end{tikzcd} \tag{4.10}$$

for pullback squares p such that $\ell_1(p) = f$ and $\ell_2(p) = [h]$. We thus add operations

$$\mathrm{src} : \mathrm{Hom} \to \mathrm{Mor} \qquad \mathrm{tgt} : \mathrm{Hom} \to \mathrm{Mor} \qquad [-] : \mathrm{Hom} \to \mathrm{Mor} \qquad \varepsilon : \mathrm{Hom} \to \mathrm{Mor};$$

in (4.10), $f = \mathrm{src}(h)$ and $g = \mathrm{tgt}(h)$. To the axioms of $\mathbb{T}_{\mathrm{LexSketch}}$, we add

$$\frac{}{\mathrm{src}(h)\downarrow} \qquad\qquad \frac{}{\mathrm{tgt}(h)\downarrow} \qquad\qquad \frac{}{[h]\downarrow}$$

$$\frac{e=\varepsilon(h,p)}{\ell_1(p)=\mathrm{src}(h) \qquad \ell_2(p)=[h] \qquad e:(\mathrm{src}(h)\circ \mathrm{pr}_1(p)) \curvearrowright \mathrm{tgt}(h)}$$

to obtain $\mathbb{T}_{\mathtt{LccSketch}}$.

(ii). The theory $\mathbb{T}_{\mathtt{LccSketch}}$ is given by the sorts of $\Sigma_{\mathtt{LccSketch}}$, its operations, the operations of $\Sigma_{\mathtt{sLexSketch}}$ and furthermore the operation

$$\mathscr{H}om : \mathrm{Mor}\times\mathrm{Mor}\to\mathrm{Hom}$$

which we think of as assigning morphisms f, g with common target an internal hom from f to g.

The axioms of $\mathbb{T}_{\mathtt{sLccSketch}}$ are given by those of $\mathbb{T}_{\mathtt{sLexSketch}}$, $\mathbb{T}_{\mathtt{LccSketch}}$ and additionally

$$\frac{h=\mathscr{H}om(f,g)}{\mathrm{src}(h)=f \qquad \mathrm{tgt}(h)=g}.$$

(iii). $\mathbb{T}_{\mathtt{sLcc}}$ extends $\mathbb{T}_{\mathtt{sLccSketch}}$ and $\mathbb{T}_{\mathtt{sLex}}$ by operations

$$\lambda_-(-,-) : \mathrm{Hom}\times\mathrm{Pb}\times\mathrm{Mor}\to\mathrm{Mor}$$

$$\mathrm{hom}_{(}\,-,-) : \mathrm{Hom}\times\mathrm{Mor}\times\mathrm{Mor}\to\mathrm{Hom}\,.$$

$\lambda_-(-,-)$ models the universal property of the evaluation maps $\varepsilon(h,p')$. If h is a local internal hom from f to g as in (4.10) and

$$\begin{tikzcd}
& x\times_s z' \arrow[dl] \arrow[dr] \arrow[rr, "e"] & & y \arrow[ddll, bend left, "g"] \\
x \arrow[dr, "f"] & p' & z' \arrow[dl] & \\
& s & &
\end{tikzcd}$$

is another diagram of the same shape, then $\lambda_h(p,e) : z'\to z$ will be the unique morphism in the slice over s that commutes with e and $\varepsilon(h,p)$.

hom has a purpose similar to that of term or pb in sLex: $h'=\mathrm{hom}_h(f,g)$ is defined if and only if f is an isomorphism with inverse g and $t(f)=s([h])$. Then, h' is an local internal hom from $\mathrm{src}(h)$ to $\mathrm{tgt}(h)$ and the evaluation maps $\varepsilon(h',p')$ are induced by the evaluation maps $\varepsilon(h,p)$ via the isomorphism f. Because we have to deal with morphism in slice categories, we introduce the partial Horn logic formula

$$e : f\curvearrowright g \equiv f=g\circ e$$

which holds if f and g have the same codomain and e is a morphism from f to g in the corresponding slice category. The additional axioms of $\mathbb{T}_{\mathtt{sLcc}}$ can now be stated as

$$\begin{array}{c} \ell_1(p)=\mathrm{src}(h) \qquad \ell_2(p)=[h] \\ \hline\hline \varepsilon(h,p)\downarrow \end{array} \qquad\qquad \frac{e=\varepsilon(h,p)}{e:(\mathrm{src}(h)\circ\mathrm{pr}_1(p))\curvearrowright g}$$

$$\frac{\mathrm{src}(h)=\mathrm{src}(h') \qquad \mathrm{tgt}(h)=\mathrm{tgt}(h') \qquad [h]=[h'] \qquad \varepsilon(h,p)=\varepsilon(h',p)}{h=h'}$$

$$\begin{array}{c} \ell_1(p) = \mathrm{src}(h) \qquad e : (\mathrm{src}(h) \circ \mathrm{pr}_1(p)) \curvearrowright \mathrm{tgt}(h) \\ \hline\hline \lambda_h(p, e) \downarrow \end{array}$$

$$\begin{array}{c} l = \lambda_h(p, e) \qquad \varepsilon(h, p') \downarrow \\ \hline l : \ell_2(p) \curvearrowright [h] \qquad \varepsilon(h, p') \circ \langle \mathrm{pr}_1(p), l \circ \mathrm{pr}_2(p) \rangle_{p'} = e \end{array}$$

$$\begin{array}{c} l = \lambda_h(p, e) \qquad l' : \ell_2(p) \curvearrowright [h] \qquad \varepsilon(h, p') \circ \langle \mathrm{pr}_1(p), l' \circ \mathrm{pr}_2(p) \rangle_{p'} = e \\ \hline l = l' \end{array}$$

$$\begin{array}{c} t(f) = t(g) \\ \hline\hline \mathscr{H}om(f, g) \downarrow \end{array} \qquad\qquad \begin{array}{c} h = \mathscr{H}om(f, g) \\ \hline \mathrm{src}(h) = f \qquad \mathrm{tgt}(h) = g \end{array}$$

$$\begin{array}{c} \mathrm{Iso}\, f\, g \qquad t(f) = s([h]) \\ \hline\hline \mathrm{hom}_h(f, g) \downarrow \end{array}$$

$$\begin{array}{c} h' = \mathrm{hom}_h(f, g) \qquad e = \varepsilon(h, p) \qquad e' = \varepsilon(h', p') \\ \hline \mathrm{src}(h') = \mathrm{src}(h) \qquad \mathrm{tgt}(h') = \mathrm{tgt}(h) \qquad [h'] = [h] \circ f \qquad e' = e \circ \langle \mathrm{pr}_1(p'), f \circ \mathrm{pr}_2(p') \rangle_p \end{array}$$

The 2-categories in the commutative square

$$\begin{tikzcd} \mathrm{sLcc} \arrow[r] \arrow[d] & \mathrm{sLccSketch} \arrow[d] \\ \mathrm{Lcc} \arrow[r] & \mathrm{LccSketch} \end{tikzcd} \tag{4.11}$$

are defined analogously to the square (4.6) for left exact categories and sketches, with two differences:

- $\mathrm{sLccSketch}_\circ$ and $\mathrm{LccSketch}_\circ$ are given by the full subcategories of $\mathrm{Mod}(\mathbb{T}_{\mathrm{sLccSketch}})$ and $\mathrm{Mod}(\mathbb{T}_{\mathrm{LccSketch}})$ of models $\mathcal{S}$ satisfying the following condition:

  $$\forall h \in \mathrm{Hom}\,\mathcal{S}\ \exists p \in \mathrm{Pb}\,\mathcal{S} :\ \varepsilon_{\mathcal{S}}(h, p) \downarrow \tag{4.12}$$

  Thus, for each $h \in \mathrm{Hom}\,\mathcal{S}$, we require that $\varepsilon_{\mathcal{S}}(h, p)$ has to be defined for some p. Note that (reducts of) models $\mathcal{C}$ of $\mathbb{T}_{\mathrm{sLcc}}$ always satisfy this condition because $\varepsilon_{\mathcal{C}}(h, \mathrm{src}(h) \times [h]) \downarrow$ for all local internal homs $h \in \mathrm{Hom}\,\mathcal{C}$.

- All 2-cells in any of the 2-categories of (4.11) are required to be *invertible*. Thus, a 2-cell $\alpha : F \Rightarrow G$ is by definition an invertible 2-cell in LinSketch.

If we allowed non-invertible 2-cells, we would not be able to construct a bi- or 2-adjunction $\mathrm{LccSketch} \leftrightarrows \mathrm{sLcc}$, essentially because the assignment $(x, y) \mapsto \mathscr{H}om(x, y)$ (where $\mathscr{H}om$ denotes the *global* internal hom here) is usually not covariant in its first argument. Consider for example the case $x_1 = \emptyset, y = \emptyset$ and x_2 an arbitrary non-empty set. Because $\mathscr{H}om(x_1, y) = \{*\}$ is non-emtpy but $\mathscr{H}om(x_2, y)$ is empty, a map

$$\mathscr{H}om(x_1, y) \to \mathscr{H}om(x_2, y) \tag{4.13}$$

induced by the unique inclusion $x_1 \to x_2$ cannot exist. If $\mathcal{S}$ is the linear sketch given by two objects a and b, then x_1, y and x_2, y induce two morphism $F_1, F_2 : \mathcal{S} \to \mathrm{Set}$. Clearly the non-invertible natural transformation $F_1 \Rightarrow F_2$ cannot be extended along the map $\mathcal{S} \to \mathcal{S}'$ to the free lcc category over $\mathcal{S}$, because this would entail the existence of a morphism (4.13).

The lower horizontal 2-functor of (4.11) is full and faithful on morphisms by definition. A proof analogous to that of 4.4.2 shows that this is also true for the upper horizontal one.

Lemma 4.5.2. *The forgetful functor* $\mathrm{sLcc}_\circ \to \mathrm{sLccSketch}_\circ$ *is full and faithful.* ☐

Denote by $\mathcal{V} : \mathrm{LccSketch} \to \mathrm{LinSketch}$ the forgetful 2-functor. Let $F_1, F_2 : \mathcal{S} \to \mathcal{C}$ be morphisms of linear sketches, with $\mathcal{S}$ an lcc sketch and $\mathcal{C}$ an lcc category such that $\mathcal{V}(F_1) = \mathcal{V}(F_2)$. It is clear from the argument before 4.4.3 that then F_1 and F_2 agree on the sorts Term and Pb. Let $h \in \mathrm{Hom}\,\mathcal{S}$. Then $F_i(h)$ is determined by

$$\mathrm{src}(F_i(h)) \qquad \mathrm{tgt}(F_i(h)) \qquad [F_i(h)] \qquad \varepsilon(F_i(h), F_i(p))$$

for $i = 1, 2$. Because this datum is independent from i, we conclude $F_1(h) = F_2(h)$, i.e. $F_1 = F_2$. Thus, morphisms $\mathcal{S} \to \mathcal{C}$ with $\mathcal{C}$ an lcc category are morphisms of linear sketches with additional properties, namely the preservation of terminal objects, pullback squares and local internal homs.

We again identify morphisms in LccSketch with codomain a strict left exact category with their image in LinSketch and say that a morphism of linear sketches is "lcc" or "strict lcc" if it is in the image of the forgetful 2-functors $\mathrm{LccSketch}(\mathcal{S}, \mathcal{C}) \to \mathrm{LinSketch}(\mathcal{S}, \mathcal{C})$ or $\mathrm{LccSketch}(\mathcal{S}, \mathcal{C}) \to \mathrm{LinSketch}(\mathcal{S}, \mathcal{C})$.

Proposition 4.5.3. *Let* $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ *be a pair of parallel lcc functors.*

(i) The strong iso-inserter $[F \overset{\sim}{\Rightarrow} G]$ *in* Cat *can be endowed with the structure of a lcc category in such a way that the projection* $P : [F \overset{\sim}{\Rightarrow} G] \to \mathcal{C}$ *is strict lcc.*

(ii) Let $\mathcal{S}$ *be a lcc sketch, and let* $H : \mathcal{S} \to [F \Rightarrow G]$ *be a morphism of linear sketches. Then*

$$H \text{ is lcc} \iff PH \text{ is lcc.}$$

If $\mathcal{S}$ *is a strict lcc sketch, then*

$$H \text{ is strict lcc} \iff PH \text{ is strict lcc.}$$

(iii) Let $\alpha : PF \Rightarrow PF$ *be the canonical natural transformation. The pair* (P, α) *is a strong inserter from* F *to* G *in* Lex*. Its universal property is preserved by the forgetful 2-functor* Lex → LexSketch*.*

If F *and* G *are strict left exact, then* (P, α) *is a strong inserter in* sLex*. Its universal property is preserved by the forgetful 2-functors* sLex → sLexSketch *and* sLex → LexSketch*.*

Proof. We have already proved the proposition with "lex" in place of "lcc" in 4.4.3. Therefore, it suffices to prove the respective preservation properties for the local internal homs.

(i). The local internal homs of $[F \overset{\sim}{\Rightarrow} G]$ are given by tuples of morphisms $h = (f, g, [h], \varepsilon)$ fitting into diagrams

$$\begin{tikzcd}
 & (w, \bar{w}) \arrow[rr, "\varepsilon"] \arrow[dl] \arrow[dr] & & (y, \bar{y}) \arrow[ddll, bend left, "g"] \\
(x, \bar{x}) \arrow[dr, "f"] & f \times [h] & (z, \bar{z}) \arrow[dl, "{[h]}"'] & \\
 & (s, \bar{s}) & &
\end{tikzcd}$$

that are mapped to diagrams of the form (4.10) in $\mathcal{C}$. Thus, the source of h is f, its target g, the underlying morphism $[h]$ and the evaluation map for the particular pullback square $f \times [h]$ is ε. A posteriori, such diagrams commute already in $[F \overset{\sim}{\Rightarrow} G]$. Let h denote the corresponding local internal hom in $\mathcal{C}$.

$\bar{z}$ and $\bar{w}$ are determined by the rest of the data. For, from the commutativity of

$$\begin{tikzcd}
Gx \times_{Gs} Gz \arrow[r, "G\varepsilon"] & Gy \\
Gx \times_{Gs} Fz \arrow[u, "\langle \mathrm{pr}_1, \bar{z} \circ \mathrm{pr}_2 \rangle"] \arrow[d, "\langle \bar{x}^{-1} \circ \mathrm{pr}_1, \mathrm{pr}_2 \rangle"'] & \\
Fx \times_{Fs} Fz \arrow[r, "F\varepsilon"] & Fy \arrow[uu, "\bar{y}"']
\end{tikzcd}$$

it follows that

$$\bar{z} = \lambda_{Gh}(q, \bar{y} \circ F\varepsilon \circ \langle \bar{x}^{-1} \circ \mathrm{pr}_1(q), \mathrm{pr}_2(q) \rangle_{Fp}).$$

Taking this equation as definition, we obtain the canonical local internal homs in $[F \overset{\sim}{\Rightarrow} G]$ which are thus uniquely determined by the fact that they are preserved by the projection $P : [F \overset{\sim}{\Rightarrow} G] \to \mathcal{C}$. The inverse of $\bar{z}$ is given by the construction with the roles of F and G reversed.

(ii) and *(iii)* are proved analogously to the corresponding statements of 4.4.3. □

Corollary 4.5.4. *The 2-functors in the diagram*

$$\begin{tikzcd}
 & \mathrm{sLex}_{2,1} \arrow[rr, "\diamond"] \arrow[dd, "\diamond"] & & \mathrm{Lex}_{2,1} \arrow[dd] \\
\mathrm{sLcc} \arrow[ur, "\diamond"] \arrow[rr, "\diamond"] \arrow[dd, "\diamond"] & & \mathrm{Lcc} \arrow[ur] \arrow[dd] & \\
 & \mathrm{sLexSketch}_{2,1} \arrow[rr, "\diamond"] & & \mathrm{LexSketch}_{2,1} \\
\mathrm{sLccSketch} \arrow[ur, "\diamond"] \arrow[rr, "\diamond"] & & \mathrm{LccSketch} \arrow[ur, "\diamond"] &
\end{tikzcd}$$

are right biadjoints. The 2-functors marked with $\diamond$ *are right 2-adjoints.*

Proof. Analogously to 4.4.4. □

Definition 4.5.5. The theory of *combinatory algebras* is given by a single sort A and three operations

$$k : A \qquad\qquad s : A \qquad\qquad (- \cdot -) : A.$$

Although $(- \cdot -)$ is usually not associative, we suppress brackets and write $a \cdot b \cdot c$ for $(a \cdot b) \cdot c$, i.e. we associate to the left. Using this convention, the axioms can be stated as

$$\frac{}{k \downarrow} \qquad\qquad \frac{}{s \downarrow} \qquad\qquad \frac{}{(a \cdot b) \downarrow}$$

$$\frac{}{k \cdot a \cdot b = a} \qquad\qquad \frac{}{s \cdot a \cdot b \cdot c = a \cdot c \cdot (b \cdot c)}$$

For the remainder of section 4.5, we specialize our meta logic to that of the effective topos.

Proposition 4.5.6. *Equality of elements in the initial combinatory algebra is undecidable.*

Proof. The initial combinatory algebra can be identified with the terms of untyped lambda calculus modulo convertibility. Convertibility of terms in untyped lambda calculus is undecidable by Rice's theorem. ☐

Proposition 4.5.7. *Let $\mathcal{C}$ be a bifree lcc category over the lcc sketch given by a single object a. Then there is an object $e \in \operatorname{Ob}\mathcal{C}$ such that $\operatorname{Hom}(e, a)$ is undecidable. In particular, $\operatorname{Mor}\mathcal{C}$ is undecidable.*

Proof. Given a method to decide equality of morphisms $\mathcal{C}$, we will construct a method for deciding equality in the free model of combinatory logic, which is impossible by 4.5.6. Informally, e will be defined as

$$e = \{(k, s, (\cdot)) \in a \times a \times a^{a \times a} \mid \forall x, y, z \in a(k \cdot x \cdot y = x \wedge s \cdot x \cdot y \cdot z = x \cdot y \cdot (x \cdot z))\}. \tag{4.14}$$

This can be made precise using the usual internal logic argument; here, e will be constructed explicitly.

Let

$$b = a \times a \times a^{a \times a} \times a \times a \times a,$$

and denote the projections, in order, by

$$k : b \to a \qquad s : b \to a \qquad (\cdot) : b \to a^{a \times a} \qquad x : b \to a \qquad y : b \to a \qquad z : b \to a.$$

If $f, g : b \rightrightarrows a$, we abbreviate

$$f \cdot g = \varepsilon \circ \langle (\cdot), f, g \rangle : Y \to a^{a \times a} \times (a \times a) \to a \tag{4.15}$$

(recall that we use ε to denote evaluation maps, and angle brackets $\langle\rangle$ for the tupling maps induced by the universal property of products or fibre products). Following the convention for combinatory algebras, we associate $\cdot$ to the left and write $f \cdot g \cdot h$ for $(f \cdot g) \cdot h$.

Let $i : c \hookrightarrow b$ be the simultaneous equalizer (i.e. limit) of the two pairs of maps

$$\left.\begin{matrix} k \cdot x \cdot y \\ x \end{matrix}\right\} : b \to a \tag{4.16}$$

and

$$\left.\begin{matrix} s \cdot x \cdot y \cdot z \\ x \cdot y \cdot (x \cdot z) \end{matrix}\right\} : b \to a.$$

Let $d = a \times a \times a^{a \times a}$. We have an evident projection $p : b \to d$ that discards the last three components corresponding to the projections x, y and z. Recall that, as in every lcc category, the pullback functor

$$p^* : \begin{cases} \mathcal{C}_{/d} \to \mathcal{C}_{/b} \\ f \mapsto \operatorname{pr}_1(p, f) \end{cases}$$

has a right adjoint $\Pi_p : \mathcal{C}_{/b} \to \mathcal{C}_{/d}$. Let $i' = \Pi_p(i) : e \hookrightarrow c$. We use k', s' and $(\cdot')$ for the obvious morphisms with domain e that factor via i through one of the projections $d \to a$ or the projection $d \to a^{a \times a}$.

We claim that the set $A = \operatorname{Hom}(e, a)$ together with its elements k', s' and the evaluation map $\cdot'$ defined analogously to (4.15) is a combinatory algebra. Only the first axiom $k' \cdot x' \cdot y' = x'$ for all $x', y' : e \to a$ will be verified here, the proof of second axiom being similar. x', y' and some irrelevant choice of morphism $e \to a$, say k', induce a morphism

$$t = \langle k', s', (\cdot'), x', y', k' \rangle : i' \curvearrowright p$$

in $\mathcal{C}_{/d}$.

By the universal property of the pullback, we obtain a morphism $t \curvearrowright p^*(i')$ in $\mathcal{C}_{/b}$, which together with the counit of the adjunction $p^* \dashv \Pi_p$ yields a morphism

$$u : t \curvearrowright p^*(i') = p^*(\Pi_p i) \curvearrowright i,$$

in $\mathcal{C}_{/d}$. Thus, we have $x' = xiu$, and similarly for y', k' and $(\cdot')$. We obtain

$$\begin{aligned} k' \cdot x' \cdot y' &= \varepsilon \circ \langle (\cdot'), \varepsilon \circ \langle (\cdot'), k', x' \rangle, y' \rangle \\ &= \varepsilon \circ \langle (\cdot), \varepsilon \circ \langle (\cdot), k, x \rangle, y \rangle \circ iu \\ &= (k \cdot x \cdot y) \circ iu \\ &= x \circ iu \\ &= x' \end{aligned}$$

by definition of i as equalizing (4.16), proving that A is a combinatory algebra.

There is a unique morphism of combinatory algebras $\phi : C \to A$, where C is the initial combinatory algebra. If we show that ϕ is injective, we are done, because then equality in C is reduced to equality in A. The category Set is locally cartesian closed. Thus, there is an lcc functor $\Psi : \mathcal{C} \to \text{Set}$ that maps a to $\operatorname{car} C$. Up to isomorphism, $\Psi(e)$ is the set (4.14) with $\operatorname{car} C$ in place of a. Clearly $(k_C, s_C, (\cdot_C)) \in \Psi(a)$. Thus, we have a map $\psi : A \to \operatorname{car} C$ given by

$$\psi : \begin{cases} A \to C \\ f \mapsto \Psi(f)(k_C, s_C, (\cdot_C)) \end{cases}$$

and it preserves k, s and $(\cdot)$, i.e. is a morphism of combinatory algebras. C is the initial combinatory algebra, so $\psi \circ \phi$ is the identity map. It follows that ϕ is injective. □

Lemma 4.5.8. *Let $\mathcal{C}$ be a category with decidable domains. Then $\operatorname{Ob} \mathcal{C}$ is a decidable set.*

Proof. For objects $x, y \in \operatorname{Ob} \mathcal{C}$, we have

$$x = y \iff (\mathrm{id}_x \circ \mathrm{id}_y) \downarrow$$

and the latter is decidable by assumption. □

Lemma 4.5.9. *Let $\mathcal{C}$ be a left exact category with decidable domains. Then $\operatorname{Mor} \mathcal{C}$ is decidable.*

Proof. Let $f, g \in \operatorname{Mor} \mathcal{C}$. Because domains and codomains of f and g are objects and hence decidable by 4.5.8 if $\mathcal{C}$ has decidable domains, we may assume that $f, g : x \rightrightarrows y$ are parallel. Then

$$f = g \iff f \circ \mathrm{id}_x = g \circ \mathrm{id}_x \iff \langle \mathrm{id}_x, \mathrm{id}_x \rangle_{f \times g} \downarrow$$

which is decidable. □

Proposition 4.5.10. *Validity for the free lcc category over the lcc sketch given by a single object is undecidable.*

Proof. Let $\mathcal{C}$ be the free lcc category over a single object. If term validity was decidable, then $\mathcal{C}$ would have decidable domains by 3.4.4 and hence $\operatorname{Mor} \mathcal{C}$ would be decidable by 4.5.9, which is absurd by 4.5.7. □

4.6 Toposes and topos sketches

Our last example are (elementary) toposes and their sketches. Toposes are left exact, cartesian closed categories with subobject classifiers. It can be proved that any such category is also lcc, so we define the theory of toposes as an extension of the theory of lcc.

Definition 4.6.1. We define theories

(i) $\mathbb{T}_{\text{TopSketch}} = (\Sigma_{\text{TopSketch}}, \mathbb{T}_{\text{TopSketch}})$ of *topos sketches*

(ii) $\mathbb{T}_{\text{sTopSketch}} = (\Sigma_{\text{sTopSketch}}, \mathbb{T}_{\text{sTopSketch}})$ of *strict topos sketches*; and

(iii) $\mathbb{T}_{\text{sTop}} = (\Sigma_{\text{sTop}}, \mathbb{T}_{\text{sTop}})$ of *strict toposes*

fitting into a commutative theory morphism diagram

$$\begin{tikzcd}
\mathbb{T}_{\text{LccSketch}} \arrow[r, hook] \arrow[d, hook] & \mathbb{T}_{\text{sLccSketch}} \arrow[r, hook] \arrow[d, hook] & \mathbb{T}_{\text{sLcc}} \arrow[d, hook] \\
\mathbb{T}_{\text{TopSketch}} \arrow[r, hook] & \mathbb{T}_{\text{sTopSketch}} \arrow[r, hook] & \mathbb{T}_{\text{sTop}}.
\end{tikzcd}$$

(i). The signature $\Sigma_{\text{TopSketch}}$ extends $\Sigma_{\text{LccSketch}}$ by a single sort

$$\text{Sub},$$

which we call the sort of *subobject classifiers*. A subobject classifier ω in $\mathcal{C}$ consists of maps

$$\text{true} = \text{true}(\omega, x) : o(x) \to [\omega]$$

for terminal objects x such that for each monomorphisms $j : u \hookrightarrow y$, there is a unique morphism $y \to [\omega]$ for which

$$\begin{tikzcd}
u \arrow[r, "!"] \arrow[d, "j"'] \arrow[dr, phantom, "p"] & o(x) \arrow[d, "\text{true}"] \\
y \arrow[r] & {[\omega]}
\end{tikzcd} \tag{4.17}$$

is a pullback square. We thus add operations

$$[-] : \text{Sub} \to \text{Ob} \qquad\qquad \text{true} : \text{Sub} \times \text{Term} \to \text{Mor}$$

To enforce the appropriate signatures of morphisms $\text{true}(\omega, x)$, we add the axioms

$$\frac{}{[\omega] \downarrow} \qquad\qquad \frac{f = \text{true}(\omega, x)}{f : o(x) \to [\omega]}$$

to $\mathbb{T}_{\text{LccSketch}}$ to obtain $\mathbb{T}_{\text{TopSketch}}$. Note that we do not put any restrictions on whether $\text{true}(\omega, x)$ is defined; this will need to be taken care of separately.

(ii). Similarly to previous cases, $\mathbb{T}_{\text{sTopSketch}}$ has an operation corresponding to some choice of subobject classifier. We thus have an additional constant symbol

$$\Omega : \text{Sub}$$

in addition to the the operations of $\mathbb{T}_{\text{TopSketch}}$ and $\mathbb{T}_{\text{sLccSketch}}$.

(iii). To model the universal property of subobject classifiers in actual toposes, we need operations

$$\chi_{-,-}(-) : \mathrm{Sub} \times \mathrm{Term} \times \mathrm{Mor} \to \mathrm{Pb}$$

so that in (4.17), $p = \chi_{\omega,x}(j)$. Similarly to previous cases, we need an operation

$$\mathrm{sub}_{-}(-,-) : \mathrm{Sub} \times \mathrm{Mor} \times \mathrm{Mor} \to \mathrm{Pb}$$

to make sure that Sub is closed under isomorphisms.

Note that $\chi_\omega(j)$ is defined if and only if j is a monomorphism. To that end, we define

$$\mathrm{Mono}\, j \equiv \mathrm{pr}_1(j \times j) = \mathrm{pr}_2(j \times j)$$

so that $\mathrm{Mono}\, j$ if and only

$$\begin{tikzcd} u \arrow[r, equal] \arrow[d, equal] & u \arrow[d, "j"] \\ u \arrow[r, "j"] & x \end{tikzcd}$$

is a pullback square, which is equivalent to j being a monomorphism.

To formalize the universal property of subobject classifiers, to assert their existence, to make Sub entirely determined by the underlying true morphisms and to close off Sub under isomorphisms, we add axioms

$$\frac{\mathrm{true}(\omega, x) = \mathrm{true}(\omega', x)}{\omega = \omega'} \qquad\qquad \frac{}{\Omega\downarrow}$$

$$\begin{array}{c} \mathrm{Mono}\, j \\ \hline\hline \chi_{\omega,x}(j)\downarrow \end{array} \qquad\qquad \frac{p = \chi_{\omega,x}(j)}{\ell_2(p) = \mathrm{true}(\omega, x) \qquad \mathrm{pr}_1(p) = j}$$

$$\frac{p = \chi_{\omega,x}(j) \qquad \ell_2(p') = \mathrm{true}(\omega, x) \qquad \mathrm{pr}_1(p') = j}{p = p'}$$

$$\begin{array}{c} \mathrm{Iso}\, f\, g \qquad t(f) = [\omega] \\ \hline\hline \mathrm{sub}_\omega(f, g)\downarrow \end{array} \qquad\qquad \frac{\omega' = \mathrm{sub}_\omega(f, g) \qquad t = \mathrm{true}(\omega, x) \qquad t' = \mathrm{true}(\omega', x)}{t = f \circ t'}$$

Analogously to the locally cartesian closed case, we define 2,1-categories

$$\begin{tikzcd} \mathrm{sTop} \arrow[r] \arrow[d] & \mathrm{sTopSketch} \arrow[d] \\ \mathrm{Top} \arrow[r] & \mathrm{TopSketch}\,. \end{tikzcd}$$

Note that the morphisms in Top are the *logical* ones (as opposed to geometric morhpisms), i.e. functors preserving all structure up to isomorphism. The 2-categories sTopSketch and TopSketch contain again not all $\mathbb{T}_{\mathbf{sTopSketch}}$ or $\mathbb{T}_{\mathbf{TopSketch}}$-models but only those models $\mathcal{S}$ that satisfy the condition (4.12) we imposed on lcc sketches and additionally

$$\forall \omega \in \mathrm{Sub}\mathcal{S} \exists x \in \mathrm{Term}\,\mathcal{S} : \mathrm{true}(\omega, x)\downarrow$$

i.e. for each subobject classifier ω, the morphism $o(x) \to \omega$ needs to be defined for some terminal object x.

We can now proceed entirely analogously to our treatment of lcc categories, and will only state the main result.

Proposition 4.6.2. *The 2-functors in the diagram*

$$\begin{tikzcd}
& \mathrm{sLcc} \arrow[rr, "\diamond"] \arrow[dd, "\diamond"] & & \mathrm{Lcc} \arrow[dd] \\
\mathrm{sTop} \arrow[ur, "\diamond"] \arrow[rr, "\diamond"] \arrow[dd, "\diamond"] & & \mathrm{Top} \arrow[ur] \arrow[dd] & \\
& \mathrm{sLccSketch} \arrow[rr, "\diamond"] & & \mathrm{LccSketch} \\
\mathrm{sTopSketch} \arrow[ur, "\diamond"] \arrow[rr, "\diamond"] & & \mathrm{TopSketch} \arrow[ur, "\diamond"] &
\end{tikzcd}$$

are right biadjoints. The 2-functors marked with $\diamond$ *are right 2-adjoints.*

The proof of proposition 4.5.7 for the undecidability of $\operatorname{Mor}\mathcal{C}$ for the free lcc category $\mathcal{C}$ relied on the fact that Set is an lcc category. But Set is also a topos, and so the same arguments show the the following statements.

Proposition 4.6.3. *Let $\mathcal{C}$ be a bifree topos over the topos sketch given by a single object a. Then there is an object $e \in \operatorname{Ob}\mathcal{C}$ such that $\operatorname{Hom}(e, a)$ is undecidable. In particular, $\operatorname{Mor}\mathcal{C}$ is undecidable.* ☐

Corollary 4.6.4. *Validity for the free topos over the topos sketch given by a single object is undecidable.* ☐

5 Conclusion

Palmgren and Vicker's partial Horn logic [17] can be used to for syntactical descriptions of free categories with additional structure. The exposition of partial Horn logic here relies not on a set of inference rules and instead interprets partial Horn logic as a convenient way for the construction of epimorphisms of finite partial algebras; the category of models is then exhibited as orthogonal subcategory to these epimorphisms. The validity checking problem is defined for arbitrary partial Horn logic theories.

Various types of categories with algebraic structure are exhibited as models of partial Horn logic theories, along with theories describing corresponding sketches. Basic bicategory theory can be used to derive from the 1-categorical freeness properties arising from theory extensions a 2- or bi-categorical one; this is especially useful when considering sketches, which provide a convenient finite presentation for potentially infinite categories with algebraic structure.

5.1 Future work

Intensional type theory as internal language for lcc infinity categories and infinity toposes has been a major focus of research for the last decade. Similarly to how there are partial Horn logic theory for lcc 1-categories and 1-toposes, it appears there are also partial Horn logic theories modelling infinity lcc 1-categories and infinity toposes, thus yielding canonical internal languages for these types of infinity categories. As equality (and hence validity) is decidable for intensional type theory, it can be expected that this holds true for infinity lcc categories, at least if an infinity category is taken to be a simplicial set with additional structure. Indeed, while infinity categories come with far more data than 1-categories, they also have to satisfy less equations on the nose.

Homotopy type theory has given rise to the study of *higher inductive types*, i.e. inductive definitions of types and their equalities at the same time. While constructions analogous to higher inductive types with non-trivial higher structure cannot exist in 1-categories, already zero-truncated higher inductive types, in which all higher structure is trivial, have been used in interesting constructions REF. These constructions do not appear to make essential use of higher structure, so there is reason to believe that analogous results can be obtained with more expressive inductive definitions 1-categories. Because partial Horn logic, or, equivalently, left exact sketches, come with free models and also allow enforcing equalities, we suggest that these may be used to extend the corresponding internal languages by additional inductive types.

On the applied side, it would be interesting to build a proof assistant based on one of the theories of 4. Clearly without elaboration, such a proof assistant would be impractical, so it should first be established which of the operations' parameters can be inferred from usage.

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