The multiverse model of dependent type theory

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Outline

Dependent type theory as essentially algebraic theory

Sorts:

$$\Gamma\ \mathrm{Ctx} \qquad \Gamma \vdash \sigma \qquad \Gamma \vdash s : \sigma \qquad f : \Gamma \to \Delta$$

$$f : (x_1 : \sigma_1, \ldots, x_m : \sigma_m) \to (y_1 : \tau_1, \ldots, y_n : \tau_n)$$
$$\iff f = (y_i \mapsto s_i \text{ term in } \Gamma)_{i=1,\ldots,n}$$

Operations/laws:

$$\frac{}{[]\ \mathrm{Ctx}} \qquad \frac{\Gamma \vdash \sigma}{\Gamma.\sigma\ \mathrm{Ctx}}$$

$$\frac{\Gamma.\sigma \vdash \tau}{\Gamma \vdash \Pi_\sigma\, \tau} \qquad \frac{\Gamma \vdash s_1 : \sigma \qquad \Gamma \vdash s_2 : \sigma}{\Gamma \vdash \mathsf{Eq}\ s_1\ s_2}$$

$$\frac{\Gamma.\sigma \vdash t : \tau}{\Gamma \vdash \lambda(t) : \Pi_\sigma\, \tau} \qquad \frac{\Gamma \vdash u : \Pi_\sigma\, \tau}{\Gamma.\sigma \vdash \mathrm{App}(t) : \tau}$$

$$\frac{f : \Gamma \to \Delta \qquad \Delta \vdash s : \sigma}{\Gamma \vdash s[f] : \sigma[f]} \qquad \frac{f : \Gamma \to \Delta \qquad g : \Delta \to E}{g \circ f : \Gamma \to E} \qquad \ldots$$

The Set model

Types are sets σ terms are elements $s \in \sigma$, parametrized over $\gamma \in \Gamma$.

- Contexts:

$$\Gamma\,\mathrm{Ctx} \iff \Gamma \in \mathrm{Set}$$

- Types:

$$\Gamma \vdash \sigma \iff (\sigma_\gamma)_{\gamma \in \Gamma} \text{ family of sets}$$

- Terms:

$$\Gamma \vdash s : \sigma \iff (s_\gamma \in \sigma_\gamma)_{\gamma \in \Gamma} \text{ family of elements}$$

- Context morphisms:

$$f : \Delta \to \Gamma \iff f \text{ is function } \Delta \to \Gamma$$

- Substitution $\Delta \vdash s[f] : \sigma[f]$ of $\Gamma \vdash s : \sigma$ along $f : \Delta \to \Gamma$:

$$(s_{f(\delta)} \in \sigma_{f(\delta)})_{\delta \in \Delta}$$

- Context extension by $\Gamma \vdash \sigma$:

$$\Gamma.\sigma = \coprod_{\gamma \in \Gamma} \sigma_\gamma = \{(\gamma, x) \mid \gamma \in \Gamma, x \in \sigma_\gamma\}$$

Generalizing the Set model

Idea: Abstract from $\mathcal{C} = \mathrm{Set}$ to arbitrary (but sufficiently nice) category.

- Contexts:

$$\Gamma\,\mathrm{Ctx} \iff \Gamma \in \mathrm{Set} \iff \Gamma \in \mathsf{Ob}\,\mathcal{C}$$

- Types:

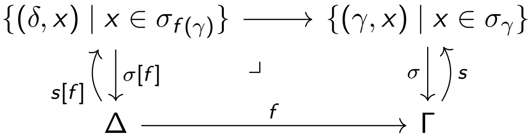
$$\begin{aligned} \Gamma \vdash \sigma &\iff (\sigma_\gamma)_{\gamma \in \Gamma} \text{ family of sets} \\ &\iff \sigma : \{(\gamma, x) \mid x \in \sigma_\gamma\} \to \Gamma \text{ function into } \Gamma \\ &\iff \sigma \in \mathsf{Ob}\,\mathcal{C}_{/\Gamma} \text{ object of slice category} \end{aligned}$$

- Terms:

$$\begin{aligned} \Gamma \vdash s : \sigma &\iff (s_\gamma \in \sigma_\gamma)_{\gamma \in \Gamma} \text{ family of elements} \\ &\iff s : \Gamma \to \mathsf{dom}\,\sigma \text{ s.t. } \sigma \circ s = \mathrm{id} \\ &\iff s : \mathrm{id}_\Gamma \to \sigma \text{ in } \mathcal{C}_{/\Gamma} \end{aligned}$$

Generalizing the Set model

- Substitution of $\Gamma \vdash s : \sigma$ along $f : \Delta \to \Gamma$:



  is pullback square. Thus:

$$\Gamma \vdash f^*(s) : f^*(\sigma)$$

- Context extension by $\Gamma \vdash \sigma$:

$$\Gamma.\sigma = \operatorname{dom} \sigma$$

- Dependent product:

$$\mathcal{C}_{/\Gamma.\sigma} \to \mathcal{C}_{/\Gamma}$$
$$(\Gamma.\sigma \vdash \tau) \mapsto (\Gamma \vdash \Pi_\sigma \tau)$$

  is right adjoint to $\sigma^* : \mathcal{C}_{/\Gamma} \to \mathcal{C}_{/\Gamma.\sigma}$, dependent sum Σ is left adjoint.

Lcc categories

Definition

A finitely complete category $\mathcal{C}$ is *locally cartesian closed (lcc)* if the pullback functor $f^* : \mathcal{C}_{/y} \to \mathcal{C}_{/x}$ has a right adjoint $\Pi_f : \mathcal{C}_{/x} \to \mathcal{C}_{/y}$ for every $f : x \to y$ in $\mathcal{C}$.

The left adjoint to f^* is $\sigma \mapsto f \circ \sigma$, always exists.

Theorem

Every lcc category can be equipped with the structure of a model of type theory.

Key for substitution stability of term and type constructors:

Lemma

Let $\mathcal{C}$ be lcc and let $x \in \mathrm{Ob}\,\mathcal{C}$. Then $\mathcal{C}_{/x}$ is lcc. If $f : x \to y$, then $f^* : \mathcal{C}_{/y} \to \mathcal{C}_{/x}$ is an lcc functor. □

Not done yet: Lcc functors preserve *up to iso*, substitution must preserve *up to equality*: Coherence problem.

Embedding lcc categories into Lcc

Denote by Lcc the $(2,1)$-category of lcc catgories, lcc functors and natural isomorphisms.

Lemma

Let $\mathcal{C}$ be an lcc category. Then the functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Lcc}_{\mathcal{C}/}$

$$x \mapsto (x^* : \mathcal{C} \to \mathcal{C}_{/x})$$
$$(f : x \to y) \mapsto (f^* : \mathcal{C}_{/y} \to \mathcal{C}_{/x})$$

is a full embedding of bicategories. □

Thus:

- If

$$\begin{array}{ccc} \mathcal{C}_{/y} & \xrightarrow{F} & \mathcal{C}_{/x} \\ & {\scriptstyle y^*}\nwarrow \quad \nearrow{\scriptstyle x^*} & \\ & \mathcal{C} & \end{array}$$

  commutes up to iso and F is lcc, then $F \cong f^*$ for some (unique) $f : x \to y$.
- If $f^* \cong g^*$ under $\mathcal{C}$, then $f = g$.

Towards the multiverse model

Model in $\mathcal{C}$, rephrased in terms of $\operatorname{Im}\mathcal{C} \subseteq \mathrm{Lcc}$:

- Contexts:

$$\Gamma\ \mathrm{Ctx} \iff \Gamma = \mathcal{C}_{/\Gamma_0} \text{ for some (unique) } \Gamma_0 \in \operatorname{Ob}\mathcal{C}$$

- Types:

$$\Gamma \vdash \sigma \iff \sigma \in \operatorname{Ob}\Gamma$$

- Terms:

$$\Gamma \vdash s : \sigma \iff s : \mathrm{id}_{\Gamma_0} \to \sigma \text{ in } \Gamma \iff s : 1 \to \sigma \text{ in } \Gamma$$

- Morphisms:

$$f : \Delta \to \Gamma \iff f : \Gamma \to \Delta \text{ lcc functor under } \mathcal{C}$$

- Substitution:

$$\Gamma \vdash s : \sigma \text{ and } \Delta \leftarrow \Gamma : f \text{ lcc} \implies \Gamma \vdash f(s) : f(\Delta)$$

- Context extension:

$$\Gamma \vdash \sigma \implies \Gamma_{/\sigma} = (\mathcal{C}_{/\Gamma_0})_{/\sigma} \cong \mathcal{C}_{/\operatorname{dom}\sigma}\ \mathrm{Ctx}$$

Are we really using that $\Gamma = \mathcal{C}_{/\Gamma_0}$ for some Γ_0?

The multiverse model

- Contexts:

$$\Gamma \,\mathrm{Ctx} \iff \Gamma \text{ lcc category}$$

- Types:

$$\Gamma \vdash \sigma \iff \sigma \in \mathrm{Ob}\,\Gamma$$

- Terms:

$$\Gamma \vdash s : \sigma \iff s : 1 \to \sigma \text{ in } \Gamma$$

- Covariant (!) context morphisms:

$$f : \Gamma \to \Delta \text{ lcc}$$

- Substitution:

$$\Gamma \vdash s : \sigma \text{ and } f : \Gamma \to \Delta \text{ lcc} \implies \Delta \vdash f(s) : f(\sigma)$$

- Context extension:

$$\Gamma \vdash \sigma \implies \Gamma.\sigma := \Gamma_{/\sigma} \,\mathrm{Ctx}$$

Coherence problems

Have, e.g.:

$$\sigma_1, \sigma_2 \in \operatorname{Ob}\Gamma \text{ and } f : \Gamma \to \Delta \text{ lcc} \implies f(\sigma_1 \times \sigma_2) \cong f(\sigma_1) \times f(\sigma_2)$$

but need equality.
$\implies$ replace Lcc by biequivalent "better" category.

Solution: A context consists of

- category $\mathcal{C}$,
- assigned choice of lcc structure on $\mathcal{C}$,
- for all lcc categories $\mathcal{D}$ with assigned lcc structure, a retraction $f \mapsto f^s$ of

$$\mathrm{sLcc}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Lcc}(\mathcal{C}, \mathcal{D})$$

  compatible with strict lcc functors in $\mathcal{D}$.

Morphisms are functors preserving assigned lcc structure and $f \mapsto f^s$ up to equality.

Model category theory: "Algebraically cofibrant algebraically fibrant lcc sketches".

Type classifiers

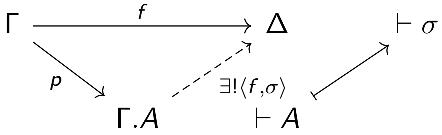
- Polymorphism: Constructions involving a *type* variable A.
- Unbounded: No need to fix universe level $A : \mathcal{U}_\ell$.
- Predicative: No type of types (no $\mathcal{U}$ or $\mathcal{U} \to \mathcal{U}$).

Definition

A model of type theory has *type classifiers* if:

$$\frac{\Gamma \ \mathrm{Ctx}}{\Gamma.A \ \mathrm{Ctx} \qquad p : \Gamma \to \Gamma.A \qquad \Gamma.A \vdash A}$$

$$\frac{f : \Gamma \to \Delta \qquad \Delta \vdash \sigma}{\langle f, \sigma \rangle : \Gamma.A \to \Delta \qquad \langle f, \sigma \rangle \circ p = f \qquad \langle f, \sigma \rangle (A) = \sigma}$$



Parametric polymorphism in the multiverse model

Lcc models rarely (never?) have type classifiers (Girard).

Theorem

The multiverse model has type classifiers. Furthermore:

$$\frac{\Gamma.A \vdash b : B \qquad \Gamma \vdash e : \sigma_1 \cong \sigma_2 \qquad \bar{\sigma}_i = \langle \mathrm{id}, \sigma_i \rangle : \Gamma.A \to \Gamma}{\Gamma \vdash e' : \bar{\sigma}_1(B) \cong \bar{\sigma}_2(B) \qquad e'(\bar{\sigma}_1(b)) = \bar{\sigma}_2(b)}$$

Substitutions of isomorphic types are naturally isomorphic.

Proof. Freely adjoin new object to lcc category. □

Open questions:

- Solve coherence problem for *dependent* type classifier:

$$\frac{\Gamma \vdash \sigma}{\Gamma.F_\sigma \ \mathrm{Ctx} \qquad p : \Gamma \to \Gamma.F_\sigma} \qquad\qquad \frac{\Gamma.F_\sigma \vdash s : p(\sigma)}{\Gamma.F_\sigma \vdash F_\sigma\, s}$$

  Should be given by freely adjoining morphism into σ.
- Type operator classifier? Modality classifier? $\rightsquigarrow$ multi logic, multi universe model.

Infinity categories

- Objects (0-cells) x, y
- Morphisms (1-cells) $f, g : x \to y$
- Homotopies (2-cells) $\alpha, \beta : f \simeq g$
- Homotopies of homotpies (3-cells) $\gamma, \delta : \alpha \simeq \beta$
- ...

All cells in dimension ≥ 2 are invertible.

Laws (e.g. associativity) hold not up to equality but homotopy one level up.

Often same things hold as for 1-categories, sometimes not. Always much more complicated.

Infinity multiverse model: Contexts are lcc ∞-categories.

Should model at least weak (*objective*?) type theory with equalities only propositional.

J elimination

$$\frac{\Gamma, x : \sigma, y : \sigma, p : \mathsf{Id}\, x\, y \vdash \tau \qquad \Gamma, z : \sigma \vdash t : \tau[x := z, y := z, p := \mathrm{refl}_z]}{\Gamma, x : \sigma, y : \sigma, p : \mathsf{Id}\, x\, y \vdash j(t) : \tau}$$

Computation rule: Substituting refl_s for p in $j(t)$ should equal (definitionally?) $t[z := s]$.

Identity types are interpreted as (homotopy) equalizer, i.e. a *limit*.

But J elimination is negative!

J elimination in the multiverse model

Context extensions $\Gamma, x : \sigma$ in multiverse model: Freely adjoin morphism $1 \to \sigma$ to lcc category Γ.

Lemma

Let Γ be an lcc ∞-category and $\sigma \in \mathrm{Ob}\,\Gamma$. Then

$$f : \qquad \Gamma, x : \sigma, y : \sigma, p : \mathrm{Id}\, x\, y \underset{z \mapsto x}{\overset{\substack{x,y \mapsto z \\ p \mapsto \mathrm{refl}_z}}{\rightleftarrows}} \Gamma, z : \sigma \qquad : g$$

is a homotopy retract of lcc ∞-categories:

$$f \circ g = \mathrm{id} \qquad\qquad \alpha : g \circ f \simeq \mathrm{id}$$

□

So:

$$j(t) : 1 \xrightarrow{g(t)} g(f(\tau)) \xrightarrow{\alpha_\tau} \tau$$

But: All computation rules hold only propositionally.

Conclusion: The multiverse model

- Contexts:
$$\Gamma\,\mathrm{Ctx} \iff \Gamma \text{ lcc}$$
- Types:
$$\Gamma \vdash \sigma \iff \sigma \in \mathrm{Ob}\,\Gamma$$
- Terms:
$$\Gamma \vdash s : \sigma \iff s : 1 \to \sigma \text{ in } \Gamma$$
- Covariant (!) context morphisms:
$$f : \Gamma \to \Delta \text{ lcc}$$
- Substitution:
$$\Gamma \vdash s : \sigma \text{ and } f : \Gamma \to \Delta \text{ lcc} \implies \Delta \vdash f(s) : f(\sigma)$$
- Context extension:
$$\Gamma \vdash \sigma \implies \Gamma.\sigma := \Gamma_{/\sigma}\,\mathrm{Ctx}$$

[Bid20] Martin E. Bidlingmaier, *An interpretation of dependent type theory in a model category of locally cartesian closed categories*, 2020.