Probabilistic Programming and Multiverse Models of Type Theory

Martin E. Bidlingmaier

PhD Dissertation

Department of Computer Science
Aarhus University
Denmark

Probabilistic Programming and Multiverse Models of Type Theory

A Dissertation
Presented to the Faculty of Natural Sciences
of Aarhus University
in Partial Fulfillment of the Requirements
for the PhD Degree

by
Martin E. Bidlingmaier

Abstract

This thesis presents two constructions in the area of mathematical semantics of type systems. First we develop a semantics of a probabilistic programming language in synthetic topology. Similarly to other branches of synthetic mathematics, synthetic topology is an axiomatic approach to topology, the study of spaces. It changes the axioms of mathematics such that ordinary sets are endowed with intrinsic topological features. Working in synthetic topology, we define a notion of distribution on an arbitrary set which takes into account the intrinsic topology. This enables us to interpret a higher-order probabilistic programming language with primitives for sampling from continuous distributions. Compared to the analytical approach, synthetic topology allows the construction of continuous distributions without having to resort to measurable spaces, which are not cartesian closed and hence cannot account for higher-order functions.

Second we develop the semantics of dependent type theory in multiverse models. In categorical semantics we are used to think of individual locally cartesian closed (lcc) categories as separate models, or universes, of dependent type theory. Instead, a multiverse model is given by a category of lcc categories and contains every small lcc category as a submodel. As in ordinary categorical semantics, there are coherence problems to be solved to make this precise. Here the multiverse approach allows the use of model category theory (in the sense of Quillen), which would otherwise be inapplicable. Using the machinery of algebraically (co)fibrant objects, we solve the coherence problems of the 1-categorical multiverse model and obtain a model of extensional type theory.

We then adapt our model categorical techniques to ∞-categories and intensional type theory. The coherence constructions enable us to interpret weak finite product and weak identity types. In contrast to the 1-categorical case, algebraically cofibrant objects are not closed under arbitrary context extensions. Nevertheless, we show that weak dependent products along base types exist.

Resumé

Denne afhandling præsenterer to konstruktioner inden for matematisk semantik af typesystemer. Først udvikler vi en semantik af et probabilistisk programmeringssprog i syntetisk topologi. På samme måde som andre grene af syntetisk matematik er syntetisk topologi en aksiomatisk tilgang til topologi, studiet af rum. Det ændrer matematikkens aksiomer, således at almindelige mængder er udstyret med iboende topologiske træk. Ved at arbejde i syntetisk topologi definerer vi et begreb om distribution på et vilkårligt sæt, som tager højde for den iboende topologi. Dette gør os i stand til at fortolke et højere ordens probabilistisk programmeringssprog med primitiver til sampling fra kontinuerlige distributioner. Sammenlignet med den analytiske tilgang tillader syntetisk topologi konstruktion af kontinuerte fordelinger uden at skulle ty til målbare rum, som ikke er kartesisk lukkede og derfor ikke kan tage højde for funktioner af højere orden.

For det andet udvikler vi semantikken for afhængig typeteori i multiversmodeller. I kategorisk semantik er vi vant til at tænke på individuelle lokalt kartesiske lukkede (lcc) kategorier som separate modeller eller universer af afhængig typeteori. I stedet er en multiversmodel givet af en kategori af lcc-kategorier og indeholder hver lille lcc-kategori som en undermodel. Som i almindelig kategorisk semantik er der sammenhængsproblemer, der skal løses for at gøre dette præcist. Her tillader multiverstilgangen brugen af modelkategori (i betydningen Quillen), som ellers ville være uanvendelig. Ved at bruge maskineriet af algebraisk (ko)fibrante objekter løser vi sammenhængsproblemerne i den 1-kategoriske multiversmodel og opnår en model for ekstensionel typeteori.

Vi tilpasser derefter vores modelkategoriske teknikker til ∞-kategorier og intensional typeteori. Sammenhængskonstruktionerne gør os i stand til at fortolke svage endelige produkt- og svage identitetstyper. I modsætning til det 1-kategoriske tilfælde lukkes algebraisk kofibrante objekter ikke under vilkårlige kontekstudvidelser. Ikke desto mindre viser vi, at der findes svage afhængige produkter langs basistyper.

Contents

Chapter 1

Overview

The thesis is composed of two loosely connected research projects: First, a semantics of *probabilistic programming* based on synthetic topology, which is the content of Chapter 2. This project was the main focus of the first half of my time as PhD-student and resulted in a joint publication on the topic with F. Faissole and B. Spitters [11]. The publication is reproduced verbatim in Chapter 2.

Second, the development of a novel categorical semantics of dependent type theory in *multiverse models*. This line of research was the main focus of the second half of my time as PhD student. So far, one paper on the 1-categorical version of the multiverse construction has been published, which is reproduced in Chapter 3. A manuscript on the adaptation to the ∞-categorical case is currently in preparation, with most results explained in Chapter 4.

This overview chapter discusses first synthetic topology and its application to probabilistic programming (Section 1.1), which may serve as an introduction to Chapter 2. We then define dependent type theory as an extension of the essentially algebraic theory of covariant cwfs (Section 1.2). Finally, we introduce the ideas leading to the multiverse model of dependent type theory (Section 1.3), which is worked out in Chapters 3 and 4.

1.1 Probabilistic programming in synthetic topology

As far as this thesis is concerned, a probabilistic program is a program which has access to a random number generator, but is otherwise pure (i.e. free of side-effect). Examples are given by the functions of listing 1.1 in ML-like syntax. The first program `is_prime` is a naive deterministic algorithm to check the primality of a given number n. The second program `is_probable_prime` is a non-trivial (i.e. non-pure) probabilistic primality test (Solovay–Strassen) which is correct with high probability.

Listing 1.1: deterministic and probabilistic primality tests

```
let is_prime n =
```

```
  if n < 2 then false
  else
    let rec check i =
      if i * i > n then true
      else (n mod i <> 0) && check (i + 1)
    in check 2

let is_probable_prime n =
  if n < 2 then false
  else if n = 2 then true
  else if n mod 2 = 0 then false
  else
    let rec check i =
      if i >= 10 then true
      else
        let a = uniform_int n in
        mod_exp n a ((n - 1) / 2) <> jacobi a n && check (i + 1)
    in check 0
```

In semantics of programming languages, one associates mathematical objects, a *meaning*, to the textual or syntactic description of programs such as those of listing 1.1. We consider semantics for the following reasons:

1. To determine when a program is correct: Does its semantics agree with the intended one? For example, we expect the semantics of the program `is_prime` of Listing 1.1 to correspond to the mathematical function $\mathbb{N} \to \{\text{true}, \text{false}\}$ given by

$$n \mapsto \begin{cases} \text{true} & \text{if } n \text{ is prime} \\ \text{false} & \text{otherwise} \end{cases}$$

2. To determine when programs are equivalent. For example, we expect that the two program fragments

```
let                          let
  x = random_int 10            x = random_int 10
  y = random_int 10            y = random_int 10
in                           in
  (x, y)                       (y, x)
```

are equivalent even though they are syntactically different (`x` and `y` are swapped in the last line). Reasoning about program equivalence is important in program optimization to determine which transformations do not change the program in unintended ways.

3. To specify mathematical objects and reason about them. Here the application of semantics techniques is in the opposite direction: We use the language and its associated semantics as a convenient way to specify the mathematical object, and the latter is our primary goal. More sophisticated computer languages even allow us to prove properties about

mathematical objects. This is exemplified by proof assistants based on dependent type theory. Here the well-typedness of a proof script, which can be verified automatically by a type checker, is equivalent to the validity of the encoded mathematical statements. An example of this technique can be found in the accompanying formalization of Chapter 2, where the proof assistant Coq is used to prove mathematical statements.

Providing a semantics of probabilistic programs such as those of Listing 1.1 is the purpose of the ALEA Coq library [6]. The approach followed in ALEA is to construct a set of sub-probability distributions $\mathfrak{G}(X)$ on a given set X such that $\mathfrak{G}$ is a monad [65] on the category of sets. This allows a deep embedding of $\mathcal{R}$ml, an extension of pure ML by primitives for random sampling from discrete probability distributions: Once the interpretations of base types such as the types are fixed (e.g. the type of natural numbers is interpreted as the set $\mathbb{N}$ of natural numbers), a program $p : \sigma \to \tau$ in $\mathcal{R}$ml is interpreted as a mathematical function $[\![p]\!] : [\![\sigma]\!] \to \mathfrak{G}([\![\tau]\!])$. Thus the semantics of a program p assigns to every element $x \in [\![\sigma]\!]$ in the interpretation of the domain σ a sub-probability distribution $[\![p]\!](x)$ on the interpretation $[\![\tau]\!]$ of the codomain. ALEA uses sub-probability distributions instead of probability distributions to account for non-terminating programs.

Concretely, an element $\mu \in \mathfrak{G}(X)$ is a function μ which assigns to each function $f : X \to [0, 1]$ to the unit interval $[0, 1] \subseteq \mathbb{R}$ a value $\mu(f) \in [0, 1]$ subject to certain conditions. A distribution is thus encoded as the integral operator on bounded real functions. Note that X is an arbitrary set, and that no continuity conditions are imposed on f. This does not pose problems for discrete distributions, where such integrals can be computed as countable sums over the carrier of the distribution. For continuous distributions on $X = \mathbb{R}$ such as the normal distribution, however, this formalism is inadequate because only Lebesgue-measurable functions are integrable.

As ALEA is tailored towards the verification of cryptographic protocols, where discrete distributions predominate, it can ignore these subtleties and stick with its simple definition of the Giry monad on sets. For other applications such as machine learning and differential privacy, however, continuous distrbutions are critical, and a solution to this problem preventing the construction of continuous distributions must be found.

The obvious approach is to define $\mathfrak{G}$ not on the category Set of sets, but on the category Meas of measurable spaces. Indeed, given a measurable space X, the set of measures on X can be endowed with the structure of a measurable space $\mathfrak{G}(X)$, and $\mathfrak{G}$ is a monad on the category measurable spaces. Our semantics of $\mathcal{R}$ml would now assign to each function $p : \sigma \to \tau$ a measurable function $[\![\sigma]\!] \to \mathfrak{G}([\![\tau]\!])$. When defining the clauses of the interpretation, one is stuck with function types, however: Function types are interpreted as exponentials, but the category Meas is not cartesian closed.

Recently, a promising approach based on quasi-Borel spaces has emerged [38]. A quasi-Borel space consists of an underlying set X together with a family M_X of functions $\mathbb{R} \to X$ subject to a number of conditions. The real numbers $\mathbb{R}$ together with the set of all measurable functions $\mathbb{R} \to \mathbb{R}$ is a quasi-Borel space on which continuous distributions can be defined. The category of QCB of quasi-Borel spaces is a quasi-topos and in particular cartesian closed, and a version of the Giry monad can be defined on QCB. Thus QCB interprets the terminating fragment of $\mathcal{R}$ml, and the construction can be adapted to potentially non-terminating programs by considering cpo objects in QCB [84].

Our alternative approach is based on *synthetic topology*. Similarly to synthetic approaches other fields, synthetic topology adapts the mathematical foundations such that all objects definable in the new foundation behave intrinsically like spaces. Thus all sets have features of topological spaces (there is a notion of open subset), and every map of sets is continuous. This is in contrast to standard topology, where topological spaces are defined as sets with additional structure, and maps between underlying sets of topological spaces need not be compatible with this structure, i.e. discontinuous.

Synthetic topology equates spaces and sets. Thus exponentials of spaces can be computed simply as sets of functions; in particular, all exponentials of spaces exist. More generally, the category of spaces is as well-behaved as the category of sets. Note that synthetic topology is not compatible with classical logic, hence we cannot assume the principle of excluded middle. Nevertheless, it is consistent to assume all of constructive logic, so that Set is a topos.

Topological spaces (in the classical sense) are not equivalent to measurable spaces, but closely related: Every topology $\mathcal{O}_X$ on a set X induces measurable space structure on X via the σ-algebra $\langle \mathcal{O}_X \rangle$ generated by the topology. Most measurable spaces of interest arise in this fashion from a topological space. A measure $\mu : \langle \mathcal{O}_X \rangle \to \mathbb{R}$ on a measurable space arising from a topology $\mathcal{O}_X$ can be identified with a function $\mathcal{O}_X \to \mathbb{R}$ subject to a number of conditions, a *valuation*, which is defined on open sets only. The proof of this requires classical logic. Constructively, it is more convenient to use valuations instead of measures.

In synthetic topology, we can associate to each set X the set $\mathfrak{G}(X) = \{\mu \mid \mu \text{ is a valuation on } X\}$, where the notion of a valuation on X is defined with respect to the intrinsic topology of X. $\mathfrak{G}$ has monad structure, which induces a model of $\mathcal{R}$ml as in ALEA, i.e. functions $p : \sigma \to \tau$ in $\mathcal{R}$ml are interpreted as mathematical functions $[\![p]\!] : [\![\sigma]\!] \to \mathfrak{G}([\![\tau]\!])$. The advantage of our Giry monad $\mathfrak{G}$ over the one of ALEA is that we can define continuous measures: One of the basic axioms of synthetic topology is that the usual metric topology on $\mathbb{R}$ coincides with the intrinsic topology of $\mathbb{R}$, which lets us define the Lebesgue valuation and distributions with a density with respect to the Lebesgue valuation.

The question arises whether a version of our model of $\mathcal{R}$ml, which is constructed based on the axioms of synthetic topology, also exists in the world of classical mathematics. This is indeed the case, although we cannot use the category of sets in classical mathematics due to the anticlassical nature of synthetic topology. Instead, we can consider a model of synthetic topology in classical mathematics. We then obtain the model of $\mathcal{R}$ml in classical mathematics in two steps: First we interpret synthetic topology in a model built using classical logic, then we interpret $\mathcal{R}$ml in this model of synthetic topology.

The main examples of models of synthetic topology are first the big topos $\mathrm{Sh}(\mathcal{T})$ (or *gros* topos) and second the realizability topos $\mathrm{RT}(\mathcal{K}_2)$ over Kleene's second algebra $\mathcal{K}_2$. The big topos is parametrized by a subcategory $\mathcal{T}$ of Top which is closed under open inclusions, and given by the sheaves over $\mathcal{T}$ with respect to open covers. Every category of sheaves over a site is a topos and hence a model of constructive logic. When $\mathcal{T}$ is (a skeleton of) the category of separable metric spaces, then in the internal logic of the big topos over $\mathcal{T}$ the real numbers are metrizable [59, Section 5.4], which is the axiom of synthetic topology needed to define continuous distributions.

The carrier of Kleene's second algebra $\mathcal{K}_2$ is Baire space $\mathbb{N}^{\mathbb{N}}$, the set of functions on natural numbers. The exponential $\mathbb{N}^{\mathbb{N}}$ in the realizability topos $\mathrm{RT}(\mathcal{K}_2)$ given by the partitioned assembly $\mathrm{id} : \mathcal{K}_2 \to \mathcal{K}_2$. Consequences of this are that $\mathcal{K}_2$ validates function choice (every surjection to $\mathbb{N}^{\mathbb{N}}$ admits a section), the continuity principle (the continuity principle (every function $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is ε-δ-continuous) and the Fan principle (every decidable bar is uniform) [87, Section 4.3]. These facts imply that $\mathbb{R}$ is metrizable in the $\mathcal{K}_2$ realizability topos [59, Section 5.3].

An advantage of our model of $\mathcal{R}$ml compared to the model in quasi-Borel spaces is that our model is based on a topos, whereas QCB is only a quasi-topos. This distinction becomes visible when we use internal languages to reason about the semantics: In synthetic topology, we may assume the principle of unique choice (that every right-unique function is the graph of a function), wheres unique choice fails in QCB. On the other hand, the Giry monad on QCB is unconditionally symmetric; in synthetic topology, we can only prove symmetry for a restricted class of well-behaved spaces (Theorem 2.27).

1.2 Dependent Type Theory

Martin-Löf's dependent type theory is the basis of modern proof assistants such as Coq, Agda, Nuprl and Lean. Dependent type theory is usually defined as a syntax generating preterms and pretypes and a set of deduction rules. The deduction rules simultaneously define subsets of well-defined types and terms, the typing relation between terms and types, and the definitional equality relation on types and terms. For the purpose of this work, dependent type theory shall instead be understood as a particular essentially algebraic theory. The advantage of our approach is that we do not have to define substitution and its behavior on variables, and that one immediately obtains a notion of model.

Partial Horn Logic. There exist a plethora of different but equivalent notions of essentially algebraic theory. The notion we shall use here is partial Horn logic [67]. Informally, a (finitary) partial Horn logic theory consists of a set of sort symbols, a set of (partial) operation symbols with assigned finite arities, and a set of axioms. Optionally we can also consider relation symbols, but these will not be needed here. Each axiom is of the form

$$\frac{\phi_1 \qquad \phi_2 \qquad \cdots \qquad \phi_m}{\psi_1 \qquad \psi_2 \qquad \cdots \qquad \psi_n} \tag{1.1}$$

where the ϕ_i and ψ_j are each of the form $t_1 = t_2$ for terms t_1, t_2. (Palmgren and Vickers [67] denote this instead by $\phi_1 \wedge \cdots \wedge \phi_m \vdash \psi_1 \wedge \cdots \wedge \psi_n$.) Terms are inductively defined depending on sort and operation symbols: There is a countable supply of variables of each sort, and operation symbols can be applied to terms if their sorts align with the the arity of the operation symbol. Every variable occurring in one of the ψ_j must occur in one of the ϕ_i.

A model X of a partial Horn logic theory consists of a carrier set X_s for each sort symbol and partial functions $p_X : X_{s_1} \times \cdots \times X_{s_n} \to X_s$ for each operation symbol p with arity $p : s_1 \times \cdots \times s_n \to s$ such that all axioms are satisfied. Here an axiom such as (1.1) holds in X if for all assignments of elements to variables such that the terms in the ϕ_i are well-defined and the equations ϕ_i hold, then also the terms in the ψ_j are well-defined and the equations ψ_j hold. More formally, this can be described as follows. Let V be the set of variables occurring in the axiom, and let $[\![v]\!] \in X_s$ be an interpretation of each variable $v \in V$ with sort s as an element of X. We extend $[\![-]\!]$ recursively to an interpretation of terms over the variables V as elements of X by $[\![p(t_1, \ldots, t_n)]\!] = p_X([\![t_1]\!], \ldots, [\![t_n]\!])$. Note that the p_X are partial functions, thus the interpretation of a given term might be undefined. We write $[\![t]\!] \downarrow$ if $[\![-]\!]$ is defined on a term t. We further extend $[\![-]\!]$ to an interpretation of equations as truth values by

$$[\![t_1 = t_2]\!] \iff [\![t_1]\!] \downarrow \wedge [\![t_2]\!] \downarrow \wedge [\![t_1]\!] = [\![t_2]\!].$$

Now X satisfies the axiom if for all interpretations $[\![-]\!]$ of the variables V as elements of X the implication

$$(\bigwedge_i [\![\phi_i]\!]) \implies (\bigwedge_j [\![\phi_j]\!])$$

holds.

A morphism $f : X \to Y$ of models consists of (total) functions $f_s : X_s \to Y_s$ for each sort symbol s which commute with the p_X and p_Y: If

$x = p_X(x_1, \ldots, x_n)$ is defined for some operation symbol p and $x_1, \ldots, x_n$ in X, then $p_Y(f(x_1), \ldots, f(x_n))$ is defined and equal to $f(x)$.

As an example, consider the theory of categories. It is given by sorts Ob of objects and Mor of morphisms. There are operation symbols $\mathrm{dom}, \mathrm{cod} : \mathrm{Mor} \to \mathrm{Ob}$ corresponding to domain and codomain of morphisms, and operations $\mathrm{id} : \mathrm{Ob} \to \mathrm{Mor}, (- \circ -) : \mathrm{Mor} \times \mathrm{Mor} \to \mathrm{Mor}$ corresponding to identity morphisms and compositions. The domain operation dom is total, which is enforced by the following axioms:

$$\frac{f = f}{\mathrm{dom}(f) = \mathrm{dom}(f)}$$

Here f is a variable of sort Mor (as is implied by its usage in the term $\mathrm{dom}(f)$). This axiom looks tautological at first, but recall that it is interpreted as follows: For all morphisms f, whenever $f = f$ holds (which is indeed trivial), then $\mathrm{dom}(f)$ is well-defined and $\mathrm{dom}(f) = \mathrm{dom}(f)$ (the latter is again trivial). Thus this axiom enforces that the dom must be interpreted as total function in all models.

There are similar axioms encoding the totality of cod and id. We henceforth denote self-equality $t = t$ of a term t by $t\downarrow$. The composition operation should be defined precisely on morphisms with compatible domain and codomain, which is encoded by the following axioms:

$$\frac{\mathrm{cod}(f) = \mathrm{dom}(g)}{(g \circ f)\downarrow} \qquad \frac{(g \circ f)\downarrow}{\mathrm{cod}(f) = \mathrm{dom}(g)}$$

Domains and codomains of identity and composed morphisms are enforced by the following axioms:

$$\frac{f = \mathrm{id}(x)}{\mathrm{dom}(f) = x \qquad \mathrm{cod}(f) = x} \qquad \frac{h = g \circ f}{\mathrm{dom}(h) = \mathrm{dom}(f) \qquad \mathrm{cod}(h) = \mathrm{cod}(g)}$$

Finally, associativity and unit laws can be encoded as follows:

$$\frac{\mathrm{cod}(f) = \mathrm{dom}(g) \qquad \mathrm{cod}(g) = \mathrm{dom}(h)}{(h \circ g) \circ f = h \circ (g \circ f)}$$

$$\frac{f\downarrow}{f = f \circ \mathrm{id}(\mathrm{dom}(f)) \qquad f = \mathrm{id}(\mathrm{cod}(f)) \circ f}$$

Covariant Categories with Families. There exist several different but more or less equivalent ways that Martin-Löf type theory can be defined as an essentially algebraic (or partial Horn logic) theory. We can pick any one of the usual notions of model of dependent type theory (comprehension categories, display map categories, categories with attributes, contextual categories, ...)

and encode its structure as partial Horn logic theory. The definition of dependent type theory we shall use here is an extension of the theory of *covariant categories with families (cwf)*, that is, categories (of contexts) equipped with a functor to the arrow category $\mathrm{Set}^{\rightarrow}$ (the type and term in context functor). For the more typical (contravariant) cwfs [27], the functor to $\mathrm{Set}^{\rightarrow}$ is contravariant, hence the opposite category functor establishes an equivalence between the categories of covariant cwfs and contravariant cwfs. The multiverse model is more naturally expressed as a covariant cwf, hence our use of this notion. Note that most authors require that cwfs have an empty contexts and are closed under context extensions; this is additional structure for our covariant cwfs.

Covariant cwfs can be encoded as models of an extension of the partial Horn logic theory of categories. In addition to the sort $\mathrm{Ctx} = \mathrm{Ob}$ of objects, or contexts, and the (context) morphisms Mor, we have sorts Ty of types and Tm of terms. Each type has an assigned context, and each term has an assigned type. We thus have operations $\mathrm{ctx} : \mathrm{Ty} \to \mathrm{Ctx}$ and $\mathrm{ty} : \mathrm{Tm} \to \mathrm{Ty}$ and axioms that enforce that both operations are total:

$$\frac{\sigma \downarrow}{\mathrm{ctx}(\sigma) \downarrow} \qquad\qquad \frac{s \downarrow}{\mathrm{ty}(s) \downarrow}$$

Here σ is a variable of sort Ty and s is a variable of sort Tm. We write $\Gamma \vdash \sigma$ for $\mathrm{ctx}(\sigma) = \Gamma$ and $\Gamma \vdash \sigma_1 = \sigma_2$ for the conjunction of $\sigma_1 = \sigma_2$ and $\mathrm{ctx}(\sigma_1) = \Gamma$. Similarly, we write $\Gamma \vdash s : \sigma$ for the conjunction of $\mathrm{ctx}(\sigma) = \Gamma$ and $\mathrm{ty}(s) = \sigma$, and $\Gamma \vdash s_1 = s_2 : \sigma$ for the conjunction of $s_1 = s_2$, $\mathrm{ty}(s) = \sigma$ and $\mathrm{ctx}(\sigma) = \Gamma$. When one of these conjunctions appear in a conclusion of an axiom, we mean several axioms with equal premises, each axiom with one of the clauses of the conjunction as conclusion.

Substitution of terms and types, i.e. functoriality along context morphisms, is encoded by operations $\mathrm{Mor} \times \mathrm{Ty} \to \mathrm{Ty}$ and $\mathrm{Mor} \times \mathrm{Tm} \to \mathrm{Tm}$, which we both ambiguously denote by function application. Substitution along morphisms $f : \Gamma \to \Delta$ is defined precisely for types and terms in context Γ. This is encoded by the following axioms:

$$\frac{f : \Gamma \to \Delta \qquad \Gamma \vdash \sigma}{\Delta \vdash f(\sigma)} \qquad\qquad \frac{f(\sigma) \downarrow}{\mathrm{ctx}(\sigma) = \mathrm{dom}(f)}$$

$$\frac{f : \Gamma \to \Delta \qquad \Gamma \vdash s : \sigma}{\Delta \vdash f(s) : f(\sigma)} \qquad\qquad \frac{f(s) \downarrow}{\mathrm{ctx}(\mathrm{ty}(s)) = \mathrm{dom}(f)}$$

Functoriality of substitution is encoded by the following axioms:

$$\frac{g(f(\sigma)) \downarrow}{(g \circ f)(\sigma) = g(f(\sigma))} \qquad\qquad \frac{g(f(s)) \downarrow}{(g \circ f)(\sigma) = g(f(s))}$$

$$\frac{\mathrm{id}(\Gamma)(\sigma) \downarrow}{\mathrm{id}(\Gamma)(\sigma) = \sigma} \qquad\qquad \frac{\mathrm{id}(\Gamma)(s) \downarrow}{\mathrm{id}(\Gamma)(s) = s}$$

We now describe extensions of the theory of covariant cwfs whose union makes up the theory of dependent type theory in its intensional and extensional variants. The two variants are distinguished by whether or not we include the equality reflection rule for equality/identity types.

The empty context. There is an empty context, i.e. an initial object $\cdot$ in the category of contexts. Thus there exists a unique morphism $! : \cdot \to \Gamma$ to every context Γ. Empty contexts are given by the following operations and axioms:

$$\frac{}{\cdot \downarrow} \qquad \frac{\Gamma \downarrow}{!_\Gamma : \cdot \to \Gamma} \qquad \frac{f : \cdot \to \Gamma}{f = !_\Gamma}$$

Context extensions. An extension of a context Γ by a variable of a given type $\Gamma \vdash \sigma$ consists of the extended context $\Gamma.\sigma$ itself, a coprojection morphism $p = p_\sigma : \Gamma \to \Gamma.\sigma$ and a variable term $\Gamma.\sigma \vdash v_\sigma : p(\sigma)$ which is initial among all such data. Thus for every $f : \Gamma \to \Delta$ and term $\Delta \vdash s : f(\sigma)$, there exists a unique context morphism $\langle f, s\rangle = \langle f, s\rangle_\sigma : \Gamma.\sigma \to \Delta$ such that $\langle f, s\rangle \circ p = f$ and $s = \langle f, s\rangle(v_\sigma)$. The existence of context extensions is enforced by the following axioms:

$$\frac{\Gamma \vdash \sigma}{p_\sigma : \Gamma \to \Gamma.\sigma \qquad \Gamma \vdash v_\sigma : p_\sigma(\sigma)}$$

$$\frac{\Gamma \vdash \sigma \qquad f : \Gamma \to \Delta \qquad \Delta \vdash s : f(\sigma)}{\langle f, s\rangle_\sigma : \Gamma.\sigma \to \Delta \qquad \langle f, s\rangle_\sigma \circ p_\sigma = f \qquad \langle f, s\rangle_\sigma(v_\sigma) = s}$$

$$\frac{g : \Gamma.\sigma \to \Delta \qquad \Delta \vdash g(v_\sigma) = s : g(\sigma)}{g = \langle f, s\rangle_\sigma}$$

Here we omitted axioms asserting that the involved operations are defined only if the obvious conditions are met. For example, if $\Gamma.\sigma$ is defined, then $\Gamma \vdash \sigma$.

If $f = \mathrm{id}$, we abbreviate $\langle f, s\rangle = \bar{s}$. When dealing with iterated context extensions, we sometimes denote the context $\Gamma.\sigma_1 \dots \sigma_n$ by $\Gamma.(x_1 : \sigma_1) \dots (x_n : \sigma_n)$ to simultaneously introduce names for the variable terms $x_i = v_{\sigma_n}$. We confusion is unlikely, we often suppress substitution along coprojection maps. Thus we write $\Gamma.\sigma_1 \dots \sigma_n \vdash x_i : \sigma_i$ as shorthand for $\Gamma.\sigma_1 \dots \sigma_n \vdash (p_{\sigma_n} \circ \cdots \circ p_{\sigma_{i+1}})(x_i) : (p_{\sigma_n} \circ \cdots \circ p_{\sigma_i})(\sigma_i)$. The map $\langle\langle f, s_1\rangle, \dots, s_n\rangle : \Gamma.\sigma_1 \dots \sigma_n \to \Delta$ induced by a morphism $f : \Gamma \to \Delta$ and appropriate terms s_i in Δ is denoted as $\langle f, s_1, \dots, s_n\rangle$. Context extensions are defined by a universal property and hence functorial. Thus if $f : \Gamma \to \Delta$ and $\Gamma \vdash \sigma$, then we obtain a map $f.\sigma = \langle p_{f(\sigma)} \circ f, v_{f(\sigma)}\rangle : \Gamma.\sigma \to \Delta.f(\sigma)$.

The unit type. There is a type $\mathrm{Unit} = \mathrm{Unit}_\Gamma$ in every context Γ and a term $\mathrm{unit} = \mathrm{unit}_\Gamma$ of type Unit:

$$\frac{\Gamma \downarrow}{\Gamma \vdash \mathrm{unit}_\Gamma : \mathrm{Unit}_\Gamma}$$

The unit type is governed by the following induction principle:

$$\frac{\Gamma.\mathrm{Unit} \vdash \sigma \qquad \Gamma \vdash s : \overline{\mathrm{unit}}(\sigma)}{\Gamma.\mathrm{Unit} \vdash \mathrm{ind}_{\mathrm{Unit}}(s) : \sigma} \qquad\qquad \frac{\Gamma.\mathrm{Unit} \vdash \sigma \qquad \Gamma \vdash s : \overline{\mathrm{unit}}(\sigma)}{\Gamma \vdash s = \overline{\mathrm{unit}}(\mathrm{ind}_{\mathrm{Unit}}(s))}$$

Thus to construct terms of a type σ depending on a variable of type Unit, it suffices to provide a term of type $\overline{\mathrm{unit}}(\sigma)$. In other words, we may assume that unit is the only term of type Unit.

Note that the induction principle as stated here is slightly abbreviated and ambiguous: The operation $\mathrm{ind}_{\mathrm{Unit}}$ depends not only on s but also σ and Γ (although Γ can be recovered as the context of σ), and its first axiom, the introduction rule, has the following converse:

$$\frac{\mathrm{ind}_{\mathrm{Unit}}(s, \sigma, \Gamma) \downarrow}{\Gamma.\mathrm{Unit} \vdash \sigma \qquad \Gamma \vdash s : \overline{\mathrm{unit}}(\sigma)}$$

Rules of this type will be omitted henceforth because they can be recovered mechanically from introduction rules: The operator depends implicitly on all variables appearing in the premise of the rule, and there is a converse to the introduction rule which states that whenever the operator is defined, then the premises of the introduction rule hold.

As all other type and term operators, the operators governing the unit type are stable under substitution, which is encoded by the following rules:

$$\frac{f : \Gamma \to \Delta}{\Delta \vdash f(\mathrm{Unit}_\Gamma) = \mathrm{Unit}_\Delta} \qquad\qquad \frac{f : \Gamma \to \Delta}{\Delta \vdash f(\mathrm{unit}_\Gamma) = \mathrm{unit}_\Delta : \mathrm{Unit}_\Delta}$$

$$\frac{f : \Gamma \to \Delta \qquad \Gamma.\mathrm{Unit}_\Gamma \vdash \sigma \qquad \Gamma \vdash s : \overline{\mathrm{unit}_\Gamma}(\sigma)}{\Delta.\mathrm{Unit}_\Delta \vdash (f.\mathrm{Unit}_\Gamma)(\mathrm{ind}_{\mathrm{Unit}}(s)) = \mathrm{ind}_{\mathrm{Unit}}(f(s))}$$

(Non-dependent) Product types. There is a binary product type whose terms are tuples:

$$\frac{\Gamma \vdash \sigma_1 \qquad \Gamma \vdash \sigma_2}{\Gamma \vdash \mathrm{Prod}\, \sigma_1\, \sigma_2} \qquad\qquad \frac{\Gamma \vdash s_1 : \sigma_1 \qquad \Gamma \vdash s_2 : \sigma_2}{\Gamma \vdash \mathrm{pair}\, s_1\, s_2 : \mathrm{Prod}\, \sigma_1\, \sigma_2}$$

It is governed by the following induction principle. Note that the coprojection $p_{\mathrm{Prod}\, \sigma_1\, \sigma_2} : \Gamma \to \Gamma.\, \mathrm{Prod}\, \sigma_1\, \sigma_2$ is abbreviated as p.

$$\frac{\Gamma.\, \mathrm{Prod}\, \sigma_1\, \sigma_2 \vdash \tau \qquad \Gamma.(v_1 : \sigma_1).(v_2 : \sigma_2) \vdash t : \langle p, \mathrm{pair}\, v_1\, v_2 \rangle(\tau)}{\Gamma.\, \mathrm{Prod}\, \sigma_1\, \sigma_2 \vdash \mathrm{ind}_{\mathrm{Prod}}(t) : \tau}$$

$$\frac{\Gamma.\, \mathrm{Prod}\, \sigma_1\, \sigma_2 \vdash \tau \qquad \Gamma.(v_1 : \sigma_1).(v_2 : \sigma_2) \vdash t : \langle p, \mathrm{pair}\, v_1\, v_2 \rangle(\tau)}{\langle p, \mathrm{pair}\, v_1\, v_2 \rangle(\mathrm{ind}_{\mathrm{Prod}}(t)) = t}$$

There is again a converse to the introduction rule of the $\mathrm{ind}_{\mathrm{Prod}}$ operator, and the following rules stating stability under substitution:

$$\frac{f : \Gamma \to \Delta \qquad \Gamma \vdash \sigma_1 \qquad \Gamma \vdash \sigma_2}{\Delta \vdash f(\mathrm{Prod}\,\sigma_1\,\sigma_2) = \mathrm{Prod}\, f(\sigma_1)\, f(\sigma_2)}$$

$$\frac{f : \Gamma \to \Delta \qquad \Gamma \vdash s_1 : \sigma_1 \qquad \Gamma \vdash s_2 : \sigma_2}{\Delta \vdash f(\mathrm{pair}\, s_1\, s_2) = \mathrm{pair}\, f(s_1)\, f(s_2) : \mathrm{Prod}\, f(\sigma_1)\, f(\sigma_2)}$$

$$\frac{\begin{array}{c} f : \Gamma \to \Delta \\ \Gamma.\,\mathrm{Prod}\,\sigma_1\,\sigma_2 \vdash \tau \qquad \Gamma.(v_1 : \sigma_1).(v_2 : \sigma_2) \vdash t : \langle p, \mathrm{pair}\, v_1\, v_2\rangle(\tau)\end{array}}{(f.\,\mathrm{Prod}\,\sigma_1\,\sigma_2)(\mathrm{ind}_{\mathrm{Prod}}(t)) = \mathrm{ind}_{\mathrm{Prod}}((f.\sigma_1.\sigma_2)(t))}$$

Equality/Identity types. Equality and identity types are where extensional and intensional type theory differ: In extensional type theory, the distinction between definitional equality (equality in the metatheory) and propositional equality (inhabitation of the equality type) is collapsed via the equality reflection rule: If the equality type of two terms is inhabited, then the two terms are definitionally equal. Intensional type theory retains many definitional equalities, but the equality reflection rule is omitted.

The canonical terms of intensional identity types are the reflexivity terms:

$$\frac{\Gamma \vdash s_1 : \sigma \qquad \Gamma \vdash s_2 : \sigma}{\Gamma \vdash \mathrm{Id}\, s_1\, s_2} \qquad\qquad \frac{\Gamma \vdash s : \sigma}{\Gamma \vdash \mathrm{refl}\, s : \mathrm{Id}\, s\, s}$$

We have the following induction principle:

$$\frac{\Gamma.(v_1 : \sigma_1).(v_2 : \sigma_2).(r : \mathrm{Id}\, v_1\, v_2) \vdash \tau \qquad \Gamma.(u : \sigma) \vdash t : \langle p, u, u, \mathrm{refl}\, u\rangle(\tau)}{\Gamma.(v_1 : \sigma_1).(v_2 : \sigma_2).(r : \mathrm{Id}\, v_1\, v_2) \vdash \mathrm{ind}_{\mathrm{Id}}(t) : \tau}$$

$$\frac{\Gamma.(v_1 : \sigma_1).(v_2 : \sigma_2).(r : \mathrm{Id}\, v_1\, v_2) \vdash \tau \qquad \Gamma.(u : \sigma) \vdash t : \langle p, u, u, \mathrm{refl}\, u\rangle(\tau)}{\langle p, u, u, \mathrm{refl}\, u\rangle(\mathrm{ind}_{\mathrm{Id}}(t)) = t}$$

Informally, the induction principle asserts that terms $r : \mathrm{Id}\, v_1\, v_2$ are generated by the reflexivity term, so that it suffices to consider the case $u = v_1 = v_2$ and $r = \mathrm{refl}\, u$ when constructing terms of a type depending on a variable of type Id. Crucially, this induction principle does not imply that every two terms of $\mathrm{Id}\, v_1\, v_2$ are equal and is thus consistent with non-trivial higher structure on types. As usual, there are rules for substitution stability:

$$\frac{f : \Gamma \to \Delta \qquad \Gamma \vdash s_1 : \sigma \qquad \Gamma \vdash s_2 : \sigma}{f(\mathrm{Id}\, s_1\, s_2) = \mathrm{Id}\, f(s_1)\, f(s_2)} \qquad\qquad \frac{f : \Gamma \to \Delta \qquad \Gamma \vdash s : \sigma}{f(\mathrm{refl}\, s) = \mathrm{refl}\ f(s)}$$

$$\frac{\begin{array}{c} f : \Gamma \to \Delta \\ \Gamma.(v_1 : \sigma).(v_2 : \sigma).(r : \mathrm{Id}\, v_1\, v_2) \vdash \tau \qquad \Gamma.(u : \sigma) \vdash t : \langle p, u, u, \mathrm{refl}\, u\rangle(\tau)\end{array}}{(f.\sigma.\sigma.\,\mathrm{Id}\, v_1\, v_2)(\mathrm{ind}_{\mathrm{Id}}(t)) = \mathrm{ind}_{\mathrm{Id}}((f.\sigma)(t))}$$

The extensional equality type is a strictly stronger version of the intensional identity type (and denoted by Eq instead of Id) because of the equality reflection rule:

$$\frac{\Gamma \vdash r : \mathrm{Eq}\, s_1\, s_2}{s_1 = s_2}$$

The axiom implies via the induction principle not only that $s_1 = s_2$ if there is a term $r : \mathrm{Eq}\, s_1\, s_2$, but also that $r = \mathrm{refl}\, s_1 = \mathrm{refl}\, s_2$.

Dependent sum types. There is a dependent sum type, whose terms are dependent pairs, i.e. pairs for which the type of the second component depends on the first component:

$$\frac{\Gamma \vdash \sigma \qquad \Gamma.\sigma \vdash \tau}{\Gamma \vdash \Sigma_\sigma\, \tau} \qquad\qquad \frac{\Gamma \vdash s : \sigma \qquad \Gamma \vdash t : \bar{s}(\tau)}{\Gamma \vdash \mathrm{dpair}\, s\, t : \Sigma_\sigma\, \tau}$$

In set theory, dependent sums correspond to sets $\{(s,t) \mid s \in \sigma, t \in \tau_s\}$ where σ is a set and $(\tau_s)_{s\in\sigma}$ is a family of sets indexed by σ. The dependent sum type is governed by the following induction principle:

$$\frac{\Gamma.\Sigma_\sigma\, \tau \vdash \kappa \qquad \Gamma.(s : \sigma).(t : \tau) \vdash k : \langle p, \mathrm{dpair}\, s\, t\rangle(\kappa)}{\Gamma.\Sigma_\sigma\, \tau \vdash \mathrm{ind}_\Sigma(k) : \kappa}$$

$$\frac{\Gamma.\Sigma_\sigma\, \tau \vdash \kappa \qquad \Gamma.(s : \sigma).(t : \tau) \vdash k : \langle p, \mathrm{dpair}\, s\, t\rangle(\kappa)}{\langle p, \mathrm{dpair}\, s\, t\rangle(\mathrm{ind}_\Sigma(k)) = k}$$

The substitution rules are as follows:

$$\frac{f : \Gamma \to \Delta \qquad \Gamma \vdash \sigma \qquad \Gamma.\sigma \vdash \tau}{f(\Sigma_\sigma\, \tau) = \Sigma_{f(\sigma)}\, (f.\sigma)(\tau)} \qquad \frac{f : \Gamma \to \Delta \qquad \Gamma \vdash s : \sigma \qquad \Gamma \vdash t : \bar{s}(\tau)}{f(\mathrm{dpair}\, s\, t) = \mathrm{dpair}\, f(s)\, f(t)}$$

$$\frac{f : \Gamma \to \Delta \qquad \Gamma.\Sigma_\sigma\, \tau \vdash \kappa \qquad \Gamma.(s : \sigma).(t : \tau) \vdash k : \langle p, \mathrm{dpair}\, s\, t\rangle(\kappa)}{(f.\Sigma_\sigma\, \tau)(\mathrm{ind}_\Sigma(k)) = \mathrm{ind}_\Sigma((f.\sigma.\tau)(k))}$$

Dependent product types. There exists a type of functions in which the type of the value of the function on some argument may depend on the argument:

$$\frac{\Gamma \vdash \sigma \qquad \Gamma.\sigma \vdash \tau}{\Gamma \vdash \Pi_\sigma\, \tau} \qquad \frac{\Gamma \vdash \sigma \qquad \Gamma.\sigma \vdash t : \tau}{\Gamma \vdash \lambda\, t : \Pi_\sigma\, \tau} \qquad \frac{\Gamma \vdash u : \Pi_\sigma\, \tau}{\Gamma.\sigma \vdash \mathrm{app}\, u : \tau}$$

$$\frac{\Gamma.\sigma \vdash t : \tau}{t = \mathrm{app}\,(\lambda\, t)} \qquad \frac{\Gamma \vdash u : \Pi_\sigma\, \tau}{u = \lambda\,(\mathrm{app}\, u)}$$

Thus terms of type $\Pi_\sigma\, \tau$ are in bijection to terms depending on a variable of type σ of type τ. In set theory, the dependent function type corresponds to the the cartesian product $\prod_{s\in\sigma} \tau_s$ of a family of sets $(\tau_s)_{s\in\sigma}$ indexed by σ. The substitution rules are as follows:

$$\frac{f : \Gamma \to \Delta \qquad \Gamma \vdash \sigma \qquad \Gamma.\sigma \vdash \tau}{f(\Pi_\sigma\, \tau) = \Pi_{f(\sigma)}\, (f.\sigma)(\tau)} \qquad \frac{f : \Gamma \to \Delta \qquad \Gamma \vdash \sigma \qquad \Gamma.\sigma \vdash t : \tau}{f(\lambda\, t) = \lambda\, ((f.\sigma)(t))}$$

$$\frac{f : \Gamma \to \Delta \qquad \Gamma \vdash u : \Pi_\sigma\, \tau}{(f.\sigma)(\mathrm{app}\, u) = \mathrm{app}\ f(u)}$$

In intensional type theory, one sometimes considers the function extensionality axiom

$$\frac{\Gamma.\sigma \vdash r : \mathrm{Id}\, t_1\, t_2}{\Gamma \vdash \mathrm{funext}\, r : \mathrm{Id}\, (\lambda\, t_1)\, (\lambda\, t_2)}$$

and an associated substitution stability axiom. The operator funext can be constructed uniquely from the equality reflection rule, hence the function extensionality axiom is superfluous in extensional type theory. For the purpose of this work, we consider function extensionality to be part of intensional type theory.

1.3 Semantics of dependent type theory in lcc categories

The relation of dependent type theory and locally cartesian closed (lcc) categories has been explored since at least Seely's seminal paper [75]. A category $\mathcal{C}$ is lcc if it is finitely complete and for each morphism $f : x \to y$ in $\mathcal{C}$, the pullback functor $f^* : \mathcal{C}_{/y} \to \mathcal{C}_{/x}$ has a right adjoint Π_f. Seely observed that the rules of extensional dependent type theory resemble the axioms of locally cartesian closed (lcc) category. To make this analogy precise, Seely gave an interpretation of the syntax of dependent type theory as objects and morphisms in a given lcc category $\mathcal{C}$. In our terminology, this amounts to defining covariant cwf structure based on $\mathcal{C}$. This covariant cwf is given as follows:

- The set of contexts is the set of objects of $\mathcal{C}$.
- The set of context morphisms is the set of morphisms in $\mathcal{C}$, but their direction is reversed: If f is a morphism in $\mathcal{C}$ from Δ to Γ, then we consider it as a (covariant) context morphisms from Γ to Δ. Thus the category of contexts and (covariant) context morphisms is given by $\mathcal{C}^{\mathrm{op}}$.
- A type in context Γ is a morphism $\sigma : \mathrm{dom}(\sigma) \to \Gamma$ with codomain Γ.
- A term of type $\sigma : \mathrm{dom}(\sigma) \to \Gamma$ is a section to σ, i.e. a map $s : \Gamma \to \mathrm{dom}(\sigma)$ such that $\sigma \circ s = \mathrm{id} : \Gamma \to \Gamma$.

- Substitution is given by pullback: If σ is a type in context Γ and $f : \Gamma \to \Delta$ is a context morphism, then $f(\sigma)$ is defined by a pullback square

$$\begin{tikzcd} \mathrm{dom}(f(\sigma)) \arrow[r] \arrow[d, "f(\sigma)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{dom}(\sigma) \arrow[d, "\sigma"] \\ \Delta \arrow[r, "f"] & \Gamma \end{tikzcd}$$

  in $\mathcal{C}$.

Conversely, Seely showed that models of dependent type theory give rise to lcc categories. Seely's correspondence extends to dependent type theories with more type formers and additional structure on lcc categories. For example, natural number types can be interpreted in lcc categories with a natural numbers objects, sum types correspond coproducts and so forth.

Unfortunately, Seely's interpretation suffers from coherence issues, which is why this covariant cwf is not well-defined: The axioms of (covariant) cwfs demand certain equalities among types which hold in the interpretation only up to isomorphism; we ignore this issue for now and will come back to this issue later.

Seely's construction allows the reduction of proof-theoretic problems to category-theoretic problems. By exhibiting suitable lcc categories, one can show that a given statement is not provable in dependent type theory, or one can show that an extension of dependent type theory by a given axiom is consistent. Lcc categories are abundant: Every elementary topos, and in particular every Grothendieck topos, is lcc. For example, the big topos of sheaves over topological spaces we discussed in Section 1.1 shows that one can consistently assume in dependent type theory that all real functions are continuous, and that Lemma of the excluded third does not follow from the axioms. The realizability topos shows that the internal Church-Turing thesis is consistent with dependent type theory, i.e. that every function of natural numbers is computable.

In the other direction, Seely's interpretation allows us to reduce problems concerning lcc categories to extensional type theory. For example, a proof of associativity of multiplication of natural numbers in dependent type theory implies via Seely's interpretation that the morphism $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ corresponding to multiplication in every lcc category with natural numbers object $\mathbb{N}$ commutes with the isomorphism $\mathbb{N} \times \mathbb{N} \cong \mathbb{N} \times \mathbb{N}$ exchanging the coordinates. This can be proved without type theory purely diagrammatically, but the proof in type theory is arguably easier to follow.

Underlying Seely's interpretation is the idea that every suitably rich (in this case: lcc) category is a separate model, or *universe*, of mathematics. Once a category $\mathcal{C}$ is fixed, the interpretation is defined in terms of data of $\mathcal{C}$ only. In Chapters 3 and 4, we explore a different point of view. Instead of regarding each $\mathcal{C}$ as a separate universe, we explore the category of all $\mathcal{C}$ as a mathematical universe. Since our mathematical universe is composed of all (small) universes, we refer to it as the *multiverse model*.

Crucial to the change of perspective that leads to the multiverse model is how we interpret contexts, i.e. what it means to declare for some type σ: *"Let x be in σ."* In the scope of this declaration, we may invoke the same deduction rules as before, but additionally there is a new constant x of type σ. We can regard this temporary change of axiom systems: The old axioms are extended by a new constant symbol x and the axiom that x is of type σ.

Proof-theoretically this may be expressed as follows. Consider a dependent type theory $\mathcal{T}$, i.e. a partial Horn logic theory which is an extension of the theory described in Section 1.2, and suppose that there is a derivation of $\cdot \vdash \sigma$. Let $\mathcal{T}'$ be the dependent type theory which is obtained from $\mathcal{T}$ by adding a new constant x in the empty context of type σ. Then there is a bijection of types and terms $(v : \sigma) \vdash t : \tau$ derivable in $\mathcal{T}$ and types and terms $\cdot \vdash t' : \tau'$ derivable in $\mathcal{T}'$.

The different contexts of dependent type theory thus parametrize a class of extensions of the underlying axiom system. If every lcc category is a separate mathematical universe, and the contexts of dependent type theory represent different axiom systems, it is natural from this point of view to interpret every context as a separate lcc category.

We should now construct a covariant cwf corresponding to the multiverse model to make our intuition precise. As Seely's original interpretation, however, our first attempt at defining this covariant cwf is not well-defined due to coherence issues: Where the axioms of type theory demand an equality of types we can only provide an isomorphism. Nevertheless, the relative simplicity of the naive construction is instructive to build intuition for the more complicated constructions needed to resolve said coherence issues.

The set of contexts of our proposed multiverse model is the set of all lcc categories. (Here and later we ignore size issues; these can be resolved by assuming sufficiently large cardinals and working with *small* lcc categories.) Let Γ be an lcc category. What are the types $\Gamma \vdash \sigma$ in the multiverse model? An arbitrary category $\mathcal{C}$ is a generalization of the category of sets, and types correspond to sets. By analogy, we define a type $\Gamma \vdash \sigma$ to be an object of Γ. The sort of all types in all contexts is thus interpreted as coproduct $\amalg_\Gamma \operatorname{Ob} \Gamma$ ranging over all lcc categories Γ.

Less obvious is how we should interpret the set of terms $\Gamma \vdash s : \sigma$: These should be interpreted as morphisms in Γ, but morphisms have a domain and a codomain, whereas a term $s : \sigma$ depends on a single type only. Note that elements of a set X are in bijection to maps $\{*\} \to X$ for some singleton set $\{*\}$. The generalization of the singleton set to a general category is a terminal object 1, which is a finite limit and thus exists in every lcc category $\mathcal{C}$. By analogy, we define a term $\Gamma \vdash s : \sigma$ for some object σ in an lcc category to be a morphism $s : 1 \to \sigma$ in $\mathcal{C}$.

Finally we have to interpret the sort of context morphisms. These must be defined such that a context morphism $f : \Gamma \to \Delta$ induces types and terms $\Delta \vdash f(s) : f(\sigma)$ for all $\Gamma \vdash s : \sigma$. The natural notion of map between categories is a functor. Objects $\Gamma \vdash \sigma$ are objects in Γ, hence we may apply a functor $f : \Gamma \to \Delta$ to σ to obtain the substitution $\Gamma \vdash f(\sigma)$. Terms $\Gamma \vdash s : \sigma$ are morphisms $s : 1 \to \sigma$, hence application of f results in a morphism $f(s) : f(1) \to f(\sigma)$. For $f(s)$ to be a term of type $f(\sigma)$ we need that $f(1)$ is the terminal object of Δ, i.e. that f preserves the terminal object. Thus we may only regard a functor $f : \Gamma \to \Delta$ as a context morphism if it preserves terminal objects. The type formers will be interpreted in terms of other structure of lcc categories, i.e. the finite limits and right adjoints to pullbacks. Type formers commute with substitution, and it will turn out that the semantical counterpart to this is that f preserves finite limits and right adjoints to pullbacks, i.e. that it is an lcc functor. We thus define a context morphisms $f : \Gamma \to \Delta$ in the multiverse model to be an lcc functor.

Next let us define context extensions in the multiverse model. Context extensions satisfy a universal property and are thus determined uniquely. Given an lcc category Γ and an object σ in Γ, we have to find an lcc category $\Gamma.\sigma$ which is obtained from Γ by freely adjoining a morphisms $v : 1 \to \sigma$. That is, lcc functors $\Gamma.\sigma \to \Delta$ have to correspond to pairs of lcc functors $f : \Gamma \to \Delta$ and morphisms $s : 1 \to f(\sigma)$ in Δ. It turns out that the slice category $\Gamma_{/\sigma}$ has the required universal property: The coprojection is given by the pullback functor $\sigma^* : \Gamma \cong \Gamma_{/1} \to \Gamma_{/\sigma}$ along the unique map $\sigma \to 1$. Slice categories of lcc categories are again lcc, and pullback functors are lcc. Thus σ^* is a context morphisms. The terminal object of $\Gamma_{/\sigma}$ is the identity on σ, so the diagonal map d in

$$\begin{tikzcd}
\sigma \arrow[r, "d"] \arrow[dr, "\mathrm{id}"'] & \sigma \times \sigma \arrow[r, "\cong"] & \operatorname{dom} \sigma^*(\sigma) \arrow[dl, "\sigma^*(\sigma)"] \\
& \sigma &
\end{tikzcd}$$

defines a term $\Gamma_{/\sigma} \vdash v : \sigma^*(\sigma)$. If $f : \Gamma \to \Delta$ is an lcc functor and $\Delta \vdash s : f(\sigma)$, then we obtain a functor

$$\Gamma_{/\sigma} \xrightarrow{f_{/\sigma}} \Delta_{/f(\sigma)} \xrightarrow{s^*} \Delta_{/1} \cong \Delta$$

under which v corresponds to s. Since every object of $\Gamma_{/\sigma}$ can be obtained as pullback along v of a map in the image of σ^*, it follows that $\Gamma_{/\sigma}$ has the required universal property of a context extension.

The type and term formers are interpreted using their categorical counterparts. For example, the unit type is interpreted as a terminal object and its induction principle via the isomorphism $\Gamma_{/1} \cong \Gamma$. Product types $\operatorname{Prod} \sigma_1\, \sigma_2$ are interpreted as categorical product $\sigma_1 \times \sigma_2$, and the induction principle is interpreted via the isomorphism $(\Gamma_{/\sigma_1})_{/\sigma_1^*(\sigma_2)} \cong \Gamma_{/\sigma_1 \times \sigma_2}$. Equality types $\operatorname{Eq} s_1\, s_2$ of terms $s_1, s_2 : \sigma$ are interpreted as equalizers of diagrams $s_1, s_2 : 1 \rightrightarrows \sigma$. Dependent sums $\Gamma \vdash \Sigma_\sigma\, \tau$ of types $\Gamma \vdash \sigma$ and $\Gamma_{/\sigma} \vdash \tau$ are given by $\operatorname{dom}(\tau)$.

Note that pullback functors f^* are right adjoints, with left adjoints Σ_f given by composition with f. Dependent sum types in the multiverse model may thus also be described via the right adjoint $\Sigma_\sigma : \Gamma_{/\sigma} \to \Gamma_{/1} \to \Gamma$, where we identify σ with the unique map $\sigma \to 1$ to the terminal object. The induction principle of dependent sum types is a reformulation of the universal property of $\Sigma_\sigma(\tau)$ that characterizes a left adjoint to σ^*. Dually, dependent product types are interpreted via the right adjoint Π_σ to pullback along σ.

The multiverse model is, to my knowledge, a novel idea in type theory. In topos theory and geometric logic, however, the idea that logic can be applied not only for constructions within individual categories, but also as a tool to relate different categories, is pervasive. Vickers [89, p. 468] writes:

> Suppose T_1 and T_2 are two geometric theories. By definition of classifying toposes, a geometric morphism $f : [T_1] \to [T_2]$ is equivalent to a model M of T_2 in $\mathcal{S}[T_1]$. Now all the objects and morphisms in $\mathcal{S}[T_1]$ are constructed out of the generic model G of T_1, and indeed can be constructed using finite limits and arbitrary colimits. It follows that M too has to be constructed out of the generic T_1-model. Let us portray this naively as a model transformation.
>
> 1. We declare "Let G be a model of T_1."
> 2. We construct a model M of T_2.
>
> Within the scope of the declaration 1, our logic and mathematics are to be interpreted in $\mathcal{S}[T_1]$ with G the generic T_1-model. This means it must be constructively valid. We thus have a temporary change of mathematics. Back outside the scope of the declaration, returning to our ambient mathematics, we find our model construction gives a geometric morphism $f : [T_1] \to [T_2]$.

Here we find explicitly the notion that introducing an indeterminate amounts to a temporary change of mathematics.

Note that Vickers writes not only about introducing a variable of a given type, but about general geometric theories. For example, we might instantiate T_1 with the theory of an indeterminate type variable, so that the type itself is the indeterminate instead of the term variable of preexisting type. In this case $\mathcal{S}[T_1]$ is the object classifier, which is given by freely adjoining a new object to the ambient topos.

The analogous construction for covariant cwfs is a context extension by an indeterminate type, a *type classifier*: Given a context Γ we would find a map $p : \Gamma \to \Gamma.T$ to a context $\Gamma.T$ and a type $\Gamma.T \vdash A$ such that (p, A) is initial among such data. Seely's semantics cannot easily account for type classifiers because the context $\Gamma.T$ must be given by an object in an lcc category $\mathcal{C}$. At most one can demand that $\mathcal{C}$ has a weak object classifiers [81], which correspond to universe types, but then one has to put size restrictions on the type variable

A to avoid Girard's paradox [19]. Furthermore, adding universe types to the type theory increases proof-theoretic strength, which can be undesirable. In our informal multiverse model, type classifiers $\Gamma.T$ can be obtained from an lcc category Γ by freely adjoining an object A to Γ. However, the construction we use to rectify the coherence issues of the multiverse model does not apply to type classifiers. have to add this somewhere We must thus leave a proper treatment of type classifiers and other more elaborate constructions for future work.

Coherence. As mentioned earlier, Seely's models and the naive multiverse model suffer from coherence issues. Consider Seely's interpretation. The first issue is that substitution of types and terms $\Gamma \vdash s : \sigma$ along context morphisms $f : \Gamma \to \Delta$ must be functorial in f. Thus if $f = \mathrm{id}$, then $f(\sigma) = \sigma$ and $f(s) = s$, and if $g : \Delta \to E$, then $g(f(\sigma)) = (g \circ f)(\sigma)$ and $g(f(s)) = (g \circ f)(s)$. Recall that Seely interprets (covariant) context morphisms $f : \Gamma \to \Delta$ as maps $f : \Delta \to \Gamma$ in an lcc category $\mathcal{C}$, with substitution given by the pullback functor f^* along f. But the assignment $f \mapsto f^*$ is only pseudo-functorial: Pullback along identity morphisms is isomorphic but not equal to the identity functor, and there is a natural isomorphism $f^*(g^*(\sigma)) \cong (g \circ f)^*(\sigma)$ if $g : \Gamma \to E$ and σ is a map to E, but we require an equality.

The second issue arises with pullback stability of type constructors. Pullback functors are indeed lcc functors and hence preserve the universal properties of objects. However, type formers are interpreted as a *choice* of universal objects, and this choice need not be preserved up to equality under pullback, only up to canonical isomorphism.

The multiverse model suffers from similar but slightly different coherence issues. Here substitution is defined by application of lcc functors, and clearly $(g \circ f)(X) = g(f(X))$ for lcc functors $\Delta \xrightarrow{f} \Gamma \xrightarrow{g} E$ and morphisms or objects X in Δ. Thus substitution *is* functorial in the multiverse model.

There is a new problem that is not present in Seely's interpretation, however. Seely interprets context extension of a type $\Gamma \vdash \sigma$, i.e. a morphism with codomain Γ, as the domain of σ. This context extension does indeed satisfy the universal property of a context extension. In the multiverse model, context extensions $\Gamma.\sigma = \Gamma_{/\sigma}$ are interpreted as slice categories. The slice category $\Gamma_{/\sigma}$ does indeed satisfy the universal property of context extensions, but only bicategorically so: Thus the map $\langle f, s\rangle : \Gamma_{/\sigma} \to \Delta$ induced by an lcc functor $f : \Gamma \to \Delta$ and a morphism $s : 1 \to f(\sigma)$ in Δ is unique up to unique natural isomorphism.

The problem with preservation of type constructors is shared among the multiverse model and Seely's models: Choices of universal objects in lcc categories are preserved not up to equality but only up to isomorphism by lcc functors and in particular by pullback functors. As far as coherence issues are concerned, the multiverse model thus shifts the problem from functoriality of substitution to the universal property of context extensions.

There exist well-known variants of Seely's construction which do not suffer from coherence issues. The idea is generally to replace the set of types in a context Γ by a suitably equivalent set such that pullbacks along maps $f : \Delta \to \Gamma$ can be constructed functorially and strictly compatible with type formers [40, 61]. There also exist syntactical methods based on rewriting [24, 25].

The multiverse model has not been considered in the context of type theory. Thus no coherence construction for this model have been considered before, and methods rectifying Seely's models are inapplicable. The development of an entirely new method is the main content of Chapters 3 (for the 1-categorical case) and 4 (for the ∞-categorical case).

Model category theory & coherence

Our coherence construction for the multiverse model makes heavy use of model category category. The notion of a model category was originally introduced by [69]. The term "model category", is short for "category of models for a homotopy theory". Thus model categories should not be confused with covariant cwfs, which are models of dependent type theory. A model category consists of an underlying complete and cocomplete category $\mathcal{M}$ and three distinguished classes of morphisms called the *fibrations*, the *cofibrations* and the *weak equivalences* subject to a number of axioms [43]. A model category $\mathcal{M}$ is meant to present the higher categorical localization $\mathcal{W}^{-1}\mathcal{M}$, where $\mathcal{W}$ is the set of weak equivalences. The classes of cofibrations and fibrations are auxiliary data that help connect 1-categorical notions in $\mathcal{M}$ to the corresponding higher categorical notions which hold in $\mathcal{W}^{-1}\mathcal{M}$.

For example, it is not generally true that a map $X \to Y$ in the localization $\mathcal{W}^{-1}\mathcal{M}$ is in the image of the functor $\mathcal{M} \to \mathcal{W}^{-1}\mathcal{M}$, but it is true if X is cofibrant and Y is fibrant. Or consider 1-categorical pushouts $B_1 \amalg_A B_2$ of spans $B_1 \xleftarrow{i_1} A \xrightarrow{i_2} B_2$ in $\mathcal{M}$. It does not generally hold that natural transformations of spans whose components are weak equivalences induce weak equivalences on pushouts. This does, however, hold if we restrict to spans of cofibrant objects for which one of i_1 or i_2 is a cofibration. In this case, the 1-categorical pushout is a homotopy pushout and satisfies a universal property also with respect to homotopies and general higher cells.

The multiverse model is an attempt at endowing a higher category (the category of lcc categories, lcc functors and natural isomorphisms) with covariant cwf structure, which is structure borne by a 1-category. Our coherence problems are thus a mismatch of 1-categorical properties demanded by type theory and the higher categorical properties which hold semantically. Since it is concerned precisely with the reduction of higher categorical to 1-categorical phenomena, model category theory is suitable framework to discuss and ultimately solve coherence problems.

Our application of model category theory can be structured in terms of the model categories we consider, each a transform and Quillen equivalent to the previous one:

1. The category Lcc **of lcc sketches.** An lcc sketch is a category equipped with sets of diagrams marked as finite limits or dependent products, without the need for these diagrams to actually satisfy the corresponding universal property. Intuitively, lcc sketches are presentations for a fully realized lcc category, much like a group can be presented using generators and relations.

The model category structure on Lcc formalizes this: The fibrant lcc sketches are the lcc categories in which precisely lcc structure is marked as such. Thus the subcategory of fibrant lcc sketches agrees with the usual category of lcc categories, and the fibrant replacement functor assigns to every lcc sketch the lcc category generated by it. Note that the lcc category generated by an lcc sketch is determined only up to contractible equivalence, hence the generation of the lcc category from a sketch can not be expressed as a 1-categorical adjunction.

A fibrant lcc sketch (i.e. lcc category) $\mathcal{C}$ "has" finite limits and dependent product in the sense that all such universal objects exist, but there is no canonical way of choosing them. The existence of these universal objects is guaranteed by lifts

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "j"'] & \mathcal{C} \\ B \arrow[ur, dashed, "b"'] & \end{tikzcd} \tag{1.2}$$

against a generating set of trivial cofibrations $j : A \to B$, which by definition must exist for fibrant $\mathcal{C}$. For example, there is a trivial cofibration j in Lcc for which A is the discrete category of two objects x_1, x_2, and B is the freestanding cospan $x_1 \leftrightarrow y \to x_2$ which is marked as supposed to be a product cone. Maps $a : A \to \mathcal{C}$ then correspond to pairs of objects in $\mathcal{C}$, and lifts b correspond to cones over the pair of objects which are marked as product cones. Other trivial cofibrations then enforce that cones marked as products indeed satisfy the universal property of products.

In detail, we construct the model category Lcc from Cat, the category of small categories with its canonical model structure, using an approach due to Isaev [47]. First we define a number of categories corresponding to the shape of universal objects of lcc categories. This induces a model category of lcc-marked categories. We then localize the model category of lcc-marked categories at a set of morphisms corresponding to the properties we wish to enforce on marked diagrams to obtain Lcc.

2. The category sLcc **of strict lcc categories.** Where the fibrant objects of Lcc are lcc sketches with the right lifting property against trivial cofibrations, the objects of sLcc are lcc sketches equipped with a canonical choice of lift b in diagrams such as (1.2). Every strict lcc categories are equipped with canonical choices of limit cone over every finite diagram and canonical right adjoints to pullback functors. The morphisms of strict lcc categories are the strict lcc functors, i.e. functors which preserve this canonical structure up to equality. The technical device in the definition of sLcc = Alg(Lcc) is the formalism of algebraically fibrant objects, which applies to a wide range of model categories. Perhaps surprisingly since not every lcc functor between strict lcc categories is isomorphic to a strict lcc functor, the model categories Lcc and sLcc are Quillen equivalent and hence present the same higher category.

As a potential model for dependent type theory, sLcc is more suitable than Lcc: Instead of having to choose, we can interpret type constructors as the *canonical* universal objects in strict lcc categories. Since the canonical universal objects are preserved up to equality by the morphisms of sLcc, the strict lcc functors, the resulting type constructors are stable under substitution.

However, context extensions $\Gamma.\sigma$ in sLcc are given by a 1-categorical pushout, which we cannot generally relate to slice categories $\Gamma_{/\sigma}$. This prevents the interpretation of dependent sum and dependent product types. For cofibrant strict lcc categories Γ there exists an equivalence $\Gamma.\sigma \simeq \Gamma_{/\sigma}$, which motivates the next model category:

3. The category Coa sLcc **of algebraically cofibrant strict lcc categories.** Where algebraically fibrant objects are objects of an underlying model category with additional data witnessing their fibrancy, algebraically cofibrant objects are equipped with data witnessing their cofibrancy. Intuitively, the objects of Coa sLcc can be understood as strict lcc categories Γ equipped with a strictification operator $F \mapsto F^s$ which assigns to each non-strict lcc functor $F : \Gamma \to \Delta$ (i.e. a functor F which preserves lcc structure up to isomorphism, but the canonical structure not up to equality) a naturally isomorphic strict lcc functor $F^s : \Gamma \to \Delta$. The morphisms of Coa sLcc are functors of underlying strict lcc categories which are compatible with the strictification operators in domain and codomain. As before, the model category Coa sLcc is Quillen equivalent to sLcc even though the underlying 1-categories are not equivalent.

The availability of the strictification operator, it turns out, is sufficient to construct an equivalences between context extensions $\Gamma.\sigma$ and slice categories $\Gamma_{/\sigma}$. Using this equivalence, we interpret dependent sum and dependent product types in addition to finite limit types, which are interpreted as before in sLcc.

Intensional type theory & ∞-categories

The equality reflection principle of extensional type theory implies that every two terms $p_1, p_2 : \mathrm{Eq}\, s_1\, s_2$ are equal, so that there exists a term of the equality type $\mathrm{Eq}\, p_1\, p_2$. Famously, Hofmann and Streicher [41] showed that the identity $\mathrm{Id}\, p_1\, p_2$ in intensional type theory need not be inhabited. This follows from the existence of the groupoid model: Types in this model are interpreted as groupoids and terms as objects of groupoids. Crucially, identity types $\mathrm{Id}\, s_1\, s_2$ of terms $s_1, s_2 : \sigma$ are interpreted as discrete groupoids of isomorphisms between s_1 and s_2 in the groupoid σ. Thus a pair of isomorphisms $p_1, p_2 : s_1 \cong s_2$ in σ such that $p_1 \neq p_2$ corresponds to terms $p_1, p_2 : \mathrm{Eq}\, s_1\, s_2$ such that $\mathrm{Id}\, p_1\, p_2$ is not inhabited.

Underlying extensional type theory is the idea that types are discrete collections of terms. As the example of groupoids shows, intensional type theory is compatible with interpreting types as higher structures: Not only can we ask *whether* terms are equal, but also the question *how* terms are equal is non-trivial. In the groupoid model, this higher structure is truncated after the first level: If $p_1, p_2 : \mathrm{Id}\, s_1\, s_2$ are two terms of an identity type, then every two terms of the iterated identity type $\mathrm{Id}\, p_1\, p_2$ are equal.

Voevodsky brought the interpretation of types as higher structures to its logical conclusion by proposing to interpret types as spaces. Thus terms are interpreted as points, and identity types as path spaces. Terms $p_1, p_2 : \mathrm{Id}\, s_1\, s_2$ of an identity type are then interpreted as paths in the underlying space σ from s_1 to s_2. Terms $q_1, q_2 : \mathrm{Id}\, p_1\, p_2$ of the iterated identity type are surfaces in σ, terms of $\mathrm{Id}\, q_1\, q_2$ are cubes in σ and so forth.

Voevodsky's interpretation can be made precise via the model of intensional type theory in the category sSet of simplicial sets [55]. Simplicial sets are a combinatorial notion of space which is often used in homotopy theory. As a model of the homotopy theory of spaces, sSet carries the structure of a model category. The interpretation of type theory in sSet is based on Awodey and Warren [7], who pointed out that the groupoid model of dependent type theory generalizes to a model in suitable model categories. One interprets the category of contexts as usual as underlying category of the model category, but only considers maps $\sigma : \mathrm{dom}\, \sigma \to \Gamma$ to an object Γ as types $\Gamma \vdash \sigma$ if σ is a fibration. In the simplicial model, this means that σ is a Kan fibration. Voevodsky's main insight was that the model in sSet has a univalent universe type $\mathcal{U}$: For types $\sigma_1, \sigma_2 : \mathcal{U}$, the type of equivalences between σ_1 and σ_2 is equivalent to the identity type $\mathrm{Id}\, \sigma_1\, \sigma_2$. *Homotopy type theory* is the extension of intensional type theory by univalent universes and higher inductive types [83].

Higher category theory has garnered significant interest in recent years, fueled especially by the theory of ∞-categories. Whereas an ordinary category is given by only objects (the 0-cells) and morphisms (the 1-cells), higher categories have a notion of n-cell for all $n \geq 0$. An *∞-category* (or, more precisely, an $(\infty, 1)$-category), is a higher category in which all n-cells for $n \geq 2$ are invertible. In contrast to general higher categories, ∞-categories are well-understood and behave in many respects similarly to 1-categories. While different formalizations of the concept exist, for us an ∞-category is a quasi-category, i.e. a simplicial set satisfying the inner Kan condition.

Intuitively we can think of ∞-categories as categories weakly enriched over spaces: For every two objects x, y, there is a mapping *space* of morphisms from x to y. All equalities one expects in ordinary category theory are replaced with coherent homotopy, e.g. the identity and associativity laws, and preservation of identity and composite morphisms by functors. The Yoneda embedding of ∞-categories is valued in the ∞-category of spaces, which plays the role that the category of sets has in ordinary category theory.

The simplicial model of intensional type theory is thus the higher analogue of the set model of extensional type theory. Similar to how the model of extensional type theory in sets generalizes to Seely's models in lcc categories, it is conjectured that intensional type theory can be interpreted in every lcc ∞-category. The converse, that every model of intensional type theory induces an lcc ∞-category, was proved by Kapulkin [54]. Extending these conjectured interpretations, it is expected that homotopy type theory has models in every elementary ∞-topos cite. Parts of these conjectures have already been proved:

- Shulman [77] proves that every *Grothendieck* ∞-topos induces a model of HoTT. Every Grothendieck ∞-topos is an elementary ∞-topos, but the converse is false. Example: Filter quotient construction due to Nima. Crucial to Shulman's proof is the fact that every Grothendieck ∞-topos is locally presentable, hence can be presented by a model category. By choosing the model category carefully, HoTT can be interpreted in the underlying 1-category of the model category.

- Kapulkin [54] shows that every model of intensional type theory induces a locally cartesian closed ∞-category, and Kapulkin and Szumiło [56] describe the internal language of finitely complete ∞-categories as intensional type theory without Π-types. The main difficulty in proving these results is that the categories are not complete and cocomplete, so model category theory is not directly applicable.

If the full conjecture was proved, it would allow us to translate problems between ∞-category theory and intensional type theory. Note that currently almost all known examples of lcc ∞-categories are in fact Grothendieck ∞-toposes, for which an interpretation is already known to exist. Arguably the lack of interest in elementary ∞-toposes is because an interpretation of type theory is not known to exist yet. For example, there has been some interest in a higher version of realizability toposes [82], which would not be a Grothendieck ∞-topos, but higher realizability toposes have so far not been studied from a purely category-theoretic perspective.

Conversely, an interpretation of HoTT in elementary ∞-toposes would allow us to conclude statements about elementary ∞-toposes from statements in HoTT. For example, the computations of fundamental groups of n-spheres in HoTT would apply to objects of n-spheres in every elementary ∞-topos. Riehl: HoTT is infty category theory for undergraduates.

As in the 1-categorical case, the naive approach to an interpretation of intensional type theory in lcc ∞-categories fails due to coherence problems. Where type theory expects equalities, we can only provide equivalences. But in addition to the problems we had with extensional type theory, the following new problems arise:

1. Associativity and identity laws in ∞-categories hold only up to homotopy. Thus there is no obvious 1-category of contexts.

2. Much like type formers, term formers are substitution stable only up to homotopy. In the 1-categorical case, term constructors are interpreted as morphisms induced by universal properties. These morphisms exist uniquely such that certain diagrams commute and are thus trivially stable under substitution. In the ∞-categorical case, such morphisms are only unique up to contractible homotopy. From this it follows that these morphisms are stable under substitution up to homotopy but not necessarily up to equality.

3. The analogue of the definitional laws that terms must satisfy, e.g. the β and η-equalities for dependent products, hold in lcc ∞-category only up to homotopy.

In Chapter 4, we discuss the ∞-categorical multiverse model. Similarly to the 1-categorical case, we wish to interpret each context as a separate lcc ∞-category, substitutions as lcc functors, types as objects and terms as morphisms with terminal domain. In the 1-categorical case the multiverse model allowed for a novel coherence construction, of which we explore an ∞-categorical adaptation. We thus construct a succession of three model categories: First the category of sketches for lcc ∞-categories, then the category of algebraically fibrant objects therein, and finally algebraically cofibrant objects. Note that the multiverse model immediately solves problem 1 above, since ∞-categories in the sense of simplicial sets satisfying the inner Kan condition are naturally organized as a model category, which has an underlying 1-category. The algebraically fibrant objects then solves the problem of substitution stable type and term formers, which are given by lifts against trivial cofibrations and are hence strictly preserved by morphisms. There is no reason to expect problem 3 to disappear, and indeed β and η rules hold only up to homotopy. These homotopies themselves are substitution stable, and so we can interpret new term formers witnessing these laws up to homotopy. We thus interpret *weak* versions of finite limit types, in which computation rules hold only up to specified homotopy. Similar relaxations of intensional type theory were considered previously [13, 85].

The third model category of algebraically cofibrant objects, allows us in the 1-categorical case to interpret dependent product and dependent sum types via an equivalence $\Gamma.\sigma \simeq \Gamma_{/\sigma}$. Crucially, algebraically cofibrant strict lcc categories are stable under context extensions in the 1-categorical case. Unfortunately, this fact does not transfer to the ∞-categorical case: Here we can only show that context extensions by variables of *base types* exist, and these variables can only be eliminated into *base terms*. Base types are types which can be described by morphisms in the category of algebraically cofibrant objects, which intuitively means that they do not arise via a type former; base terms are defined similarly. Base types and terms are a purely semantical phenomenon as they cannot be part of the syntax by definition.

Chapter 2

Probabilistic programming

Abstract

The ALEA Coq library formalizes measure theory based on a variant of the Giry monad on the category of sets. This enables the interpretation of a probabilistic programming language with primitives for sampling from discrete distributions. However, continuous distributions have to be discretized because the corresponding measures cannot be defined on all subsets of their carriers.

This paper proposes the use of synthetic topology to model continuous distributions for probabilistic computations in type theory. We study the initial σ-frame and the corresponding induced topology on arbitrary sets. Based on these intrinsic topologies we define valuations and lower integrals on sets, and prove versions of the Riesz and Fubini theorems. We then show how the Lebesgue valuation, and hence continuous distributions, can be constructed.

2.1 Introduction

Monads on Cartesian closed categories are a semantics for a large class of effectful functional programming languages [65]. The ALEA Coq library [6] provides an interpretation of $\mathcal{R}$ml, a functional programming language with primitives for random choice, by constructing a version of the Giry monad [35] on the category of Coq's types. Giry monads generally assign to a suitable class of spaces their spaces of measures or valuations, and in ALEA's case it is the class of discrete spaces. ALEA's monad is suitable for embedding programming languages with discrete sampling constructs into the ambient logic of Coq, as for example in applications to cryptography [9]. But continuous distributions are essential in statistics, machine learning and differential privacy, and these distributions have to be discretized in ALEA because they cannot be defined on discrete spaces. For example, the Lebesgue measure is only defined on Borel sets, and hence is not directly definable in ALEA.

We propose the use of *synthetic topology* [59] as a principled way of resolving the problem of continuous distributions. In synthetic topology, one works in constructive (in our case even predicative) mathematics to which one adds axioms that make sets behave much like topological spaces. The precise mathematical foundation we have in mind is *Homotopy Type Theory (HoTT)* as it is used in modern proof assistants, on top of which we assume the necessary axioms of synthetic topology.

HoTT has a number of advantages over standard intensional type theory, even when one is only interested in sets, i.e. types with trivial higher structure. ALEA can only prove its version of the Giry monad to adhere to the monad laws pointwise and resorts to setoids because neither function extensionality nor quotients are part of standard Coq. This is not a problem in HoTT, where function extensionality is provable and quotients of sets can be constructed as a special case of higher inductive types. We refer to Section 2.2 for a detailed discussion of our mathematical foundations.

Our main contribution is a development of the theory of valuations (which play the role of measures) and lower integrals on sets in synthetic topology. We show that a version of the Riesz theorem holds in this setting: Valuations are in one-to-one correspondence with lower integrals. This is then used to define a Giry monad $\mathfrak{G}$ on the category of sets in terms of the continuation monad, and we prove a version of the Fubini theorem. Assuming the *metrizability* of the real numbers $\mathbb{R}$, which asserts that the intrinsic topology on the set $\mathbb{R}$ agrees with the metric topology, we then define the Lebesgue valuation as an element of $\mathfrak{G}(\mathbb{R})$. Finally, we obtain an interpretation of $\mathcal{R}$ml, a call-by-value PCF with probabilistic effects, via the restriction of the Giry monad $\mathfrak{G}$ to sub-probability valuations.

In non-classical measure theory (which is required because the metrizability of $\mathbb{R}$ is contradictory with classical logic), the Dedekind or Cauchy real numbers have to be replaced by the lower reals $\mathbb{R}_l$ because the former are not closed under enumerable suprema. A lower real is a lower closed rounded inhabited subset of $\mathbb{Q}$, and in synthetic topology it is natural to require that this subset is furthermore an open subset. An analogous construction for Dedekind reals in synthetic topology is studied by Lešnik [59] in great generality. The HoTT book [83] also proposes this in the special case of S equal to the initial σ-frame, and a formalization inspired by Coq's Math Classes [78] using the HoTT library [8] has been carried out by Gilbert [34]. We develop the theory of lower reals valued in the initial σ-frame and construct an isomorphism $\mathbb{R}_l \cong \mathbb{Q}_\omega$ with the ω-cpo completion of the rationals $\mathbb{Q}$.

The initial σ-frame is itself the ω-cpo completion of the partial order of booleans $\perp \leq \top$, or, equivalently, the pointed ω-cpo completion of the unit set $1 = \{*\}$. Pointed ω-cpo completions of sets are studied by Altenkirch et al. [2] in HoTT using quotient inductive inductive types [3]. We explain how their construction can be adapted to ω-cpo completions of preorders with respect to covers. This generality is needed to define ω-cpo completions of the rationals and the definition of a formal σ-frame of opens in the Dedekind reals $\mathbb{R}$.

In concurrent work with our initial work on this topic [31], Huang [44] developed the semantics of a probabilistic programming language targeted at machine learning with semantics in topological domains. Meanwhile, Huang et al. [45] have connected the two approaches by showing that the interpretation of a valuation in the internal logic of the K2-realizability topos indeed gives the notion of valuation on topological domains as defined by Huang [44].

Some of the results presented in this paper are formalized[1] in Coq on top of the HoTT library. The core of the formalization consists of a proof of the dominance property of the Sierpinski space (Theorem 2.7), most of our discussion of the lower reals (Section 2.5), and the definition of the Giry monad (Definition and Proposition 2.25).

The paper is structured as follows. Section 2.2 contains some of the order-theoretic preliminaries and notation used throughout the paper. Section 2.3 discusses the construction and properties of ω-cpo completions. Section 2.4 studies the initial σ-frame as a set of truth values in synthetic topology. Section 2.5 constructs the lower reals and contains a proof of their universal property (Theorem 2.11). Section 2.6 defines valuations and integrals and proves their equivalence (Theorem 2.17, a variant of the Riesz theorem). Section 2.7 constructs the Giry monad and proves a variant of the Fubini theorem (Theorem 2.27). Section 2.8 discusses the metrizability of $\mathbb{R}$ and constructs the Lebesgue valuation. Section 2.9 provides an interpretation of $\mathcal{R}$ml based on the Giry monad that can account for continuous distributions. Section 2.10 concludes.

2.2 Preliminaries

Logical foundations Our logical foundation is predicative and constructive mathematics. *Constructivity* means that we do not assume classical principles such as the lemma of the excluded middle, and that we do not assume the existence of choice functions. *Predicativity* means that we do not use the powerset construction, i.e. we do not assume that there are sets $\mathcal{P}(A) = \{B \mid B \subseteq A\}$.

The concrete system that we have in mind is *Homotopy Type Theory (HoTT)*, and specifically its theory of *(homotopy) sets*, i.e. types with trivial higher structure. Rijke and Spitters [73] prove that the category of sets in HoTT form a ΠW-pretopos, which is the category theoretic description of our logical foundation.

HoTT's inductive types allow the construction of effective quotients. Effectivity means that the principle of unique choice holds: If for a binary relation $R \subseteq X \times Y$ we have that for all $x \in X$ there exists a unique $y \in Y$ such that $R(x, y)$, then there exists a function $f_R : X \to Y$ such that $R(x, f_R(x))$ for

[1] `https://github.com/FFaissole/Valuations/tree/d06d2c8c9cce3ddf6137`

all x. Furthermore, functions in HoTT satisfy function extensionality: If two functions agree pointwise, then the two functions are equal.

These two principles are fairly unusual for systems based on type theory, and they are not provable in Agda or Coq. While some of HoTT's higher structure, in particular the univalence axiom, can make it more convenient to work even with hsets, higher types are not strictly required for our work. We are thus optimistic that our work could in principle be formalized in systems such as XTT [80] or OTT [1] that promise to extract HoTT's well-behaved logic of hsets while discarding higher principles such as the univalence axiom for hsets. However, not all quotients in XTT and OTT are well-behaved in the sense that they satisfy the principle of unique choice, hence our reliance on it might obstruct such a formalization project.

Predicative foundations reject the notion of a subobject classifier (i.e. a set Ω of truth values such that every subset of a set X corresponds to a map $X \to \Omega$), but they permit universes of small, bounded sets. We thus assume a countable hierarchy of universes $\mathcal{U}_0 \subseteq \mathcal{U}_1 \subseteq \dots$ of small sets. Universes allow the definition of sets of small proposition $\Omega_0 \subseteq \Omega_1 \subseteq \dots$ by restriction to sets with at most one element. We thus obtain small powersets $\mathcal{P}_i(X)$ as sets of functions $X \to \Omega_i$. The bookkeeping of the current universe/subset level i is essentially trivial; we will thus simply write Ω to mean the set Ω_i for a fixed level i where confusion is unlikely.

While all existing systems that implement our intended logical foundations are based on type theory, we stick to the usual set theoretic notation in this paper. We emphasize that the difference between our set theoretic notation and type theory is only superficial: For example, when we write $X \subseteq Y$, we mean that there is an evident injective coercion from X to Y. A set comprehension $\{x \in X \mid \phi(x)\}$ corresponds to the dependent sum type $\Sigma_{x \in X}\phi(x)$ and is used only when $\phi(x)$ is a proposition for all x. We adopt the convention that the phrase "there exists" refers to a proof-relevant, i.e. untruncated statement; when we mean the proof irrelevant notion we say that something "*merely*" exists. Similarly, we write $X \to Y$ for the set of functions from X to Y here; elsewhere, this set is often denoted by Y^X.

In addition to the logical foundations based on HoTT's hsets, we require two additional principles to work with synthetic topology. First, we assume the existence of *free ω-cpo completions* (Assumption 1). As explained in Section 2.3, this is a fairly weak assumption; it follows from a number of other common axioms (impredicativity, HoTT's quotient-inductive-inductive types or the axiom of countable choice).

Second and more invasive, we assume that the set of real numbers is *metrizable* (Assumption 2). The metrizability axiom asserts that the intrinsic topology (see Section 2.4) on $\mathbb{R}$ agrees with the usual metric topology. It contradicts classical logic, and is not satisfied in most models of our logic, i.e. ΠW-pretoposes. However, it does hold in the big topos of topological spaces and in the K2 realizability topos (see Section 2.8).

Semantics As we work internally in a rather unusual logic, we have to justify its consistency, in particular the consistency of the two axioms that we assume on top of predicative mathematics. First of all, a model of predicative mathematics is a ΠW-pretopos [73]. Every topos is, in particular, a ΠW-pretopos, and so our results apply to the internal logic of every topos that validates our axioms. [59] proves that both the K2 realizability topos and the topological toposes [32, 86] validate our two assumptions and can thus interpret all of our constructions.

The topos used in Fourman [32, 33] and the topos of continuous M-actions for the localic monoid of endomorphisms of Baire space used in Van Der Hoeven and Moerdijk [86] are equivalent by the Comparison Lemma [50, Theorem C.2.2.3] because the topological monoid M is dense in the site of separable locales, all of which can be covered by Baire space. Thus sheaves in the latter topos can be seen as a uni-typed version of sheaves in the former topos. Both of these works provide a constructive elaboration of Brouwer's continuity principles.

Our specific realization of predicative mathematics is the theory of 0-truncated types, *sets*, in HoTT. This, however, poses new semantical problems: ΠW-pretoposes and in particular 1-toposes do not model HoTT's higher dimensional priniciples such as the univalence axiom.

While we expect that both the K2 realizability topos and the topological toposes can be suitably embedded into models of HoTT, only the case of topological toposes appears to be resolved (though see e.g. Swan and Uemura [82] for progress on realizability models): It was proved by Shulman [77] that most of HoTT as presented in the HoTT book can be interpreted in all Grothendieck ∞-toposes. Shulman's ∞-topos models can also interpret propositional resizing (impredicativity), and so Assumption 1 holds in these models. Every Grothendieck 1-topos of sheaves over a finitely complete site is equivalent to the category of 0-truncated objects in the corresponding ∞-topos [5]. In particular, this holds for sheaves over the site of small topological spaces, thus the model of HoTT in this ∞-topos also validates Assumption 2.

Order theory Let us review some basic notions from order and domain theory. A *preorder* consists of a carrier set P and a transitive and reflexive relation $x \leq y$ on P. We generally identify a preorder with its carrier set P, leaving the order relation implicit. A map $f : P \to Q$ of preorders is *monotone* if $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in P$. A *partial order* is a preorder whose ordering relation is antisymmetric. A *suborder* of a partial order P is a monotone map $i : P' \hookrightarrow P$ with P' a partial order such that $i(x) \leq i(y)$ implies $x \leq y$. Suborders of P may be identified with subsets of P.

Let I and P be preorders and let $d : I \to P$ be a monotone map. The *join* $\bigvee d = \bigvee_{i \in I} d(i)$ of d is a least element such that $d(i) \leq \bigvee d$ for all $i \in I$. Dually, a *meet* $\bigwedge d = \bigwedge_{i \in I} d(i)$ is a greatest element such that $d(i) \geq \bigwedge d$ for all $i \in I$. Joins and meets are uniquely determined up to isomorphism, i.e. if e and e' are both joins (or both meets) of the same diagram D, then $e \leq e'$ and $e' \leq e$. If we say that certain kinds of joins or meets exist in a preorder, we mean that there is a function that assigns to every suitable diagram its join or meet, respectively, and these canonical joins and meets are denoted by $\bigvee -$ and $\bigwedge -$. If P is a partial order, then joins and meets are unique if they *merely* exist, and so by unique choice we obtain unique join and meet functions.

Identifying subsets $U \subseteq P$ with suborders of P, we write $\bigvee U \in P$ for the join over the corresponding inclusion map. A monotone map $f : I' \to I$ of preorders I' and I is *final* if for each $i \in I$ there merely exists $i' \in I'$ such that $i \leq f(i')$. If $d : I \to P$ is a monotone map into a partial order P and $f : I' \to I$ is final, then the two joins $\bigvee d$ and $\bigvee (d \circ f)$ exist and agree if either one exists.

A preorder I is *directed* if I is inhabited and there is a function $u : I \times I \to I$ (not necessarily monotone) such that for all $i, j \in I$ we have $i \leq u(i,j)$ and $j \leq u(i,j)$. The partial order ω has for its carrier set the natural numbers with its natural order (which is generated by $n \leq n+1$ for all n). If I is enumerable (i.e. there exists a surjection $\mathbb{N} \twoheadrightarrow I$) and directed, then there exists a final map $\omega \to I$. Thus enumerable directed joins in partial orders P can be reduced to joins over maps $\omega \to P$, i.e. chains $x_0 \leq x_1 \leq \dots$ in P.

Bottom and *top* elements are joins $\bot = \bigvee \emptyset$ and meets $\top = \bigwedge \emptyset$, respectively, over the empty set. A *lattice* is a partial order L which has all binary joins $x \vee y = \bigvee\{x,y\}$ and binary meets $x \wedge y = \bigwedge\{x,y\}$ for $x, y \in L$. It is *distributive* if $x \wedge (y \vee z) = (x \wedge y) \vee (y \wedge z)$ holds for all $x, y, z \in L$. An *ω-complete partial order (ω-cpo)* is a partial order which has all enumerable directed joins. A monotone map $f : C \to D$ of ω-cpos C and D is *ω-(Scott-)continuous* if f preserves enumerable directed joins. A *σ-frame* is a partial order with bottom and top elements, binary meets and enumerable joins which satisfy the distributivity law $x \wedge \bigvee_{n \in \mathbb{N}} y_n = \bigvee_{n \in \mathbb{N}} (x \wedge y_n)$. A partial order P is a σ-frame if and only if it has top and bottom elements and is both a distributive lattice and an ω-cpo: Arbitrary enumerable joins can be computed as $\bigvee_{n \in \mathbb{N}} x_n = \bigvee_{n \in \omega} (x_0 \vee \cdots \vee x_n)$ using just the lattice and ω-cpo structure.

Sets of truth values $\Omega = \Omega_i$ are partially ordered by implication. They are stable under joins (disjunctions) and meets (conjunctions) over small indexing sets.

2.3 Presentations of ω-cpos

In this section we adapt the notion of *dcpo presentation* described in Jung et al. [53] for ω-cpo presentations. We discuss three proofs of the existence of free ω-cpo completions, and construct presentations of product ω-cpos.

Definition 2.1. An *ω-cpo presentation* consists of a preorder P and a *cover relation* $\triangleleft \subseteq P \times \mathcal{P}(P)$ such that $p \triangleleft U$ (p is *covered* by U) holds only if U is an enumerable directed suborder of P (thus U is given by a map $\mathbb{N} \to P$ with directed image). We generally leave the covering relation $\lhd$ implicit and refer to the ω-cpo presentation $(P, \lhd)$ as just P. A morphism of *ω-cpo presentations* $f : P \to Q$ is a monotone map preserving covers, in the sense that if $p \lhd U$ holds in P, then $f(p) \lhd f(U)$ holds in Q for all $p \in P$ and $U \subseteq P$.

Every ω-cpo C can be regarded as an ω-cpo presentation with cover relation

$$c \lhd U \iff c \leq \bigvee U$$

for $U \subseteq C$ directed and enumerable. ω-continuous maps $C \to D$ of ω-cpos may be identified with their morphisms when considered as ω-cpo presentations.

Assumption 1. Let P be an ω-cpo presentation. Then there is a free ω-cpo over P, i.e. there is a morphism $\eta : P \to P_\omega$ of ω-cpo presentations with P_ω an ω-cpo such that for any given morphism $f : P \to C$ with C an ω-cpo there is a unique ω-continuous map $\bar{f} : P_\omega \to C$ such that $\bar{f}\eta = f : P \to C$.

It appears that Assumption 1 is independent of constructive predicative mathematics. However, it follows from rather weak additional mathematical principles, all of which are generally considered constructive.

As a first option, one can work with propositional resizing (impredicativity) [83], i.e. assume that the inclusions $\Omega_0 \subseteq \Omega_1 \subseteq \ldots$ are equalities. Working impredicatively, Jung et al. [53] construct free dcpos over dcpo presentations. We sketch a straightforward adaptation of their proof for ω-cpos. Say a lower subset $\mathfrak{a} \subseteq P$ is an *ideal* if from $p \lhd U$ and $U \subseteq \mathfrak{a}$ it follows that $p \in \mathfrak{a}$. Let $\mathrm{Idl}(P)$ be the partial order of all ideals. Ideals are closed under arbitrary intersections, so every subset $M \subseteq P$ is contained in the least ideal containing it:

$$\langle M \rangle = \bigcap \{\mathfrak{a} \in \mathrm{Idl}(P) \mid M \subseteq \mathfrak{a}\}.$$

It follows that $\mathrm{Idl}(P)$ has all joins and that they can be computed as $\bigvee_{i \in I} \mathfrak{a}_i = \langle \bigcup_{i \in I} \mathfrak{a}_i \rangle$. Assigning to each $p \in P$ the principal ideal $\langle \{q \in P \mid q \leq p\} \rangle$ gives a monotone map from P to $\mathrm{Idl}(P)$ which preserves covers. It exhibits $\mathrm{Idl}(P)$ as the free suplattice over P, i.e. the free partial order with all joins subject to the cover relations. Now P_ω can be defined as the least subset of $\mathrm{Idl}(P)$ which contains the principal ideals that is closed under joins of enumerable directed families.

Next, P_ω can be constructed as a quotient inductive inductive type (QIIT) [3] in homotopy type theory. The special case of the free ω-cpo with bottom element over a set (i.e. discrete partial order without covers) is worked out in Altenkirch et al. [2]. Given a set A, they define $A_\perp$ and a dependent predicate $\leq : A_\perp \times A_\perp \to \Omega$ mutually recursive as a QIIT. Elements of $A_\perp$ and their equalities are generated by the constructors

$$\eta : A \to A_\perp \qquad \bigvee : \left(\sum_{x:\mathbb{N}\to A_\perp} \prod_{n:\mathbb{N}} x_n \leq x_{n+1} \right) \to A_\perp$$

$$\perp : A_\perp \qquad \alpha : \prod_{x,y:A_\perp} x \leq y \to y \leq x \to x = y.$$

$\leq$ has constructors corresponding to reflexivity, transitivity and the universal properties of $\perp$ and $\bigvee$. The recursion principle for $A_\perp$ as QIIT is the universal property of the free domain over A. This argument can easily be adapted for our purpose: To construct P_ω given an ω-cpo presentation P, one omits from the scheme defining $P_\perp$ the constructor $\perp$ and adds constructors $\prod_{p,q:P} p \leq q \to \eta(p) \leq \eta(q)$ corresponding to monotonicity of η and

$$\prod_{p:P} \prod_{U \in \mathcal{P}(P)} p \lhd U \to \eta(p) \leq \bigvee c_U$$

where $c_U : \mathbb{N} \to P$ is a monotone and final map into U. The semantics of QIITs are not entirely understood, but it is proved in Lumsdaine and Shulman [60] that all Grothendieck ∞-topos models validate the existence of many HITs. Work on reducing QIITs to such simpler inductive constructions is ongoing; see [3].

As a third alternative, P_ω can be constructed as a quotient of the set $\mathrm{Hom}(\omega, P)$ of monotone sequences in P if one is willing to assume the axiom of countable choice, at least in the important special case where the covering relation is such that $p \lhd U$ holds only if $u \leq p$ for all $u \in U$, which is true in all our applications. A similar construction for $A_\perp$ is worked out in Altenkirch et al. [2], with the general idea going back to Rosolini [74]. Let $\leq'$ be the preorder on the set of monotone functions $\mathrm{Hom}(\omega, P)$ which is generated from $c \leq' d$ if for all n there merely exists m such that $c_n \leq d_m$, and $\eta(p) \leq' c_U$ whenever $p \lhd U$, where $\eta(p)$ denotes the constant sequence with value p and c_U is a final sequence in U. If $c, d : \omega \to P$ are monotone and $c \leq' d$, then it can be shown by induction over transitivity of $\leq'$ that for all m, n there merely exist either m' or n' such that $c(m')$ respectively $d(n')$ is an upper bound for both $c(m)$ and $d(n)$. It follows that the image of the set-theoretic transpose $\bar{c} : \mathbb{N} \times \mathbb{N} \to P$ of a monotone function $c : \omega \to \mathrm{Hom}(\omega, P)$ ($\bar{c}$ need not be monotone with respect to the product order) is directed: The *mere* existence of binary upper bounds implies the existence of a function assigning upper bounds because of the bijection $\mathbb{N} \times \mathbb{N} \cong \mathbb{N}$ and countable choice. We obtain a final sequence $c' : \omega \to P$, which can be shown to be a join of c. Let P_ω be the quotient partial order of the preorder $(\mathrm{Hom}(\omega, P), \leq')$. By countable choice, every sequence $c : \omega \to P_\omega$ can be lifted to one in $\mathrm{Hom}(\omega, P)$, where its join can be computed and mapped back to P_ω. Thus P_ω is an ω-cpo, and the verification of its universal property is straightforward.

Proposition 2.2. *The free ω-cpo completion is monotone on functions: If $f \le g : P \to Q$, then $f_\omega \le g_\omega : P_\omega \to Q_\omega$.*

Proof. The subset $\{x \in P_\omega \mid f_\omega(x) \le g_\omega(x)\}$ contains $\eta(p)$ for all $p \in P$ and is closed under directed enumerable joins. $\square$

Jung et al. [53, proposition 2.8] construct presentations of product dcpos based on presentations of their factors, and an analogous result holds for ω-cpos. Our proof differs slightly from theirs because we do not assume that ω-completions are constructed as sets of ideals and instead rely solely on the universal property.

Proposition 2.3. *Let P and Q be ω-cpo presentations. Define a cover relation on the product partial order $P \times Q$ by $(p,q) \triangleleft U \times \{q\}$ if $p \triangleleft U$ in P and $(p,q) \triangleleft \{p\} \times V$ if $q \triangleleft V$ in Q. Then the canonical map $f : (P \times Q)_\omega \to P_\omega \times Q_\omega$ is an order isomorphism.*

Proof. Let $g_0 : P \to (Q \to (P \times Q)_\omega)$ be the function assigning to each $p \in P$ the function $q \mapsto \eta(p,q)$. The set of functions $Q \to (P \times Q)_\omega$ is an ω-cpo with joins computed pointwise. If $p \triangleleft U$ and $q \in Q$, then $\bigvee_{u \in U} g_0(u)(q) = \bigvee \eta(U) \times \{q\} \ge \eta(p,q) = g_0(p,q)$ by definition of the cover relation on $P \times Q$. Thus g_0 preserves covers and induces an ω-continuous map $g_1 : P_\omega \to (Q \to (P \times Q)_\omega)$. Let $g_2 : Q \to (P_\omega \to (P \times Q)_\omega)$ be its transpose; it is valued in ω-continuous functions. Suppose $q \triangleleft V$ and let us prove that for each $x \in P_\omega$ we have

$$g_2(q)(x) \le \bigvee_{v \in V} g_2(v)(x). \tag{2.1}$$

If $x = \eta(p)$ for some $p \in P$, then this holds because $(p,q) \triangleleft \{p\} \times V$ in $P \times Q$. If (2.1) holds for every element $x \in W$ for a directed enumerable family $W \subseteq P_\omega$, then

$$g_2(q)(\bigvee W) = \bigvee_{x \in W} g_2(q)(x) \le \bigvee_{x \in W} \bigvee_{v \in V} g_2(v)(x) = \bigvee_{v \in V} g_2(v)(\bigvee W)$$

because $g_2(q)$ and $g_2(v)$ for all v commute with joins and joins commute among each other. Thus g_2 preserves covers and induces an ω-continuous map $g_3 : Q_\omega \to (P_\omega \to (P \times Q)_\omega)$. Let $g : P_\omega \times Q_\omega \to (P \times Q)_\omega$ be its transpose.

g is ω-continuous in each argument. Thus if $p : I \to P_\omega$ and $q : I \to Q_\omega$ are monotone maps with I enumerable and directed, then

$$g(\bigvee_{i \in I} (p_i, q_i)) = \bigvee_{i \in I} \bigvee_{j \in I} g(p_i, q_j) = \bigvee_{k \in I} g(p_k, q_k)$$

because, I being directed, the diagonal $I \to I \times I$ is final. It follows that g is ω-continuous. Thus gf is the identity by the universal property of the ω-cpo completion, and $fg = \mathrm{id}$ holds by the universal property of products. $\square$

Corollary 2.4. *Let P be an ω-cpo presentation. If P has a bottom element $\bot$, then $\eta(\bot) \in P_\omega$ is a bottom element, and likewise for top elements. If P has all binary joins which are compatible with covers in the sense that $\vee : P \times P \to P$ preserves the covers on $P \times P$ defined in Proposition 2.3, then P_ω has all binary joins and $\eta : P \to P_\omega$ preserves them. The same is true for binary meets.*

Proof. Without loss of generality, we may assume that for all $p \in P$ we have $p \lhd \{p\}$ because adding these covers to P does not change the generated ω-cpo P_ω. Endow the terminal partial order 1 with the covering relation $* \lhd \{*\}$, where $* \in 1$ is the unique element of the unit set. Then the map $P \to 1$ is a map of ω-cpo presentations, and so are its right or left adjoints $1 \to P$ if they exist. Because $1_\omega = 1$ and the ω-cpo completion is monotone (Proposition 2.2), it follows that $P_\omega \to 1$ is a right (left) adjoint if $P \to 1$ is. Thus P_ω has a bottom (top) element if P has one.

Suppose $p \lhd U$ in P. Then

$$(\eta(p), \eta(p)) \leq \bigvee_{u \in U} \bigvee_{v \in U} (\eta(u), \eta(v)) = \bigvee_{w \in U} (\eta(w), \eta(w))$$

because U is directed. We may thus add the diagonal covers

$$(p, p) \lhd \{(u, u) \mid u \in U\} \tag{2.2}$$

to the covers of $P \times P$ without changing the generated ω-cpo. Because $P \times P$ presents the product $P_\omega \times P_\omega$, the diagonal $P_\omega \to P_\omega \times P_\omega$ is obtained by ω-cpo completion of the diagonal of P. Now suppose P has binary joins which preserve the covers defined in Proposition 2.3. Binary joins will always preserve diagonal covers as in (2.2). Thus the binary join map can be extended to a left adjoint to the diagonal of P_ω, i.e. P_ω has binary joins. Similarly, if P has a cover preserving binary meet map, then its extension to P_ω will be right adjoint to the diagonal. $\square$

2.4 Synthetic topology and the initial σ-frame

In synthetic topology [28, 46, 59] one works with sets and functions as if they behave like topological spaces and continuous maps. For this analogy to have any value, the very least one would expect is a notion of *open subset* of a given set (i.e. space). The set of (small) subsets of a given set A is given by the set of functions $A \to \Omega$. It is thus natural to expect a subset $S \subseteq \Omega$ that classifies the *open* subsets, in the sense that a function $A \to \Omega$ is the indicator function of an open subset if and only if it factors via S. S may be thought of as the *set of open truth values*. We obtain sets $\mathcal{O}(A) = (A \to S)$ of open subsets for every set (space) A, and it can indeed be verified that the preimage of an open subset under every function is again open. Thus all functions are continuous.

In traditional (analytic) topology, S corresponds to the *Sierpinski space*: The space with carrier Ω whose only nontrivial open is the singleton set $\{\top\}$.

Indicator functions $\chi : A \to \Omega$ with A a topological space (in the usual sense) are continuous if and only if the preimage of $\top$ is open; in other words if and only if χ corresponds to an open subset.

Without imposing any further requirements on S, there is not much we can say about the sets $\mathcal{O}(A)$. For example, $S = \emptyset$ might be empty, in which case only the empty subset has any open subsets at all. If $S = \{\top\}$, then $\mathcal{O}(A) = \{A\}$ for all A. For $S = \mathbb{B} = \{\bot, \top\}$ the booleans, the opens are precisely the decidable subsets. In this case, S is closed under finite conjunctions and disjunction, corresponding to open subsets being closed under finite intersections and unions. But in constructive models, the booleans are usually not closed under infinite conjunction, so we may not assume that any infinite unions of opens are open. Arguably the most interesting case is where S is a proper subset of Ω (so that the topology is not discrete), contains the boolean truth values $\top$ and $\bot$ and is closed under *enumerable* disjunction. This makes it possible to study limits and first-countable spaces such as the real numbers, which are at the heart of integration theory. Following the HoTT book and Gilbert [34], we take for S the least subset of Ω satisfying these constraints: The initial σ-frame.

Definition and Proposition 2.5 ([34]). *The* Sierpinski space $\mathbb{S} = \mathbb{B}_\omega$ *is the free ω-cpo over the partial order $\mathbb{B} = \{\bot \leq \top\}$ of decidable truth values. $\mathbb{S}$ admits the structure of a σ-frame, and it is the initial one. The map $\mathbb{S} \to \Omega$ given by $s \mapsto s = \top$ exhibits $\mathbb{S}$ as a suborder of Ω and preserves all σ-frame structure.*

Thus $\mathbb{S}$ is a suborder of Ω, and we freely identify elements $s \in \mathbb{S}$ with their image in Ω. The preservation of enumerable joins by the inclusion $\mathbb{S} \subseteq \Omega$ means that if $\bigvee_{n \in \mathbb{N}} s_n = \top$ holds for an enumerable family of elements $s_n \in \mathbb{S}$, then there merely exists n such that $s_n = \top$.

As explained in Section 2.3, in the presence of countable choice $\mathbb{S}$ may be identified with monotone binary sequences $\omega \to \mathbb{B}$, where we distinguish sequences only by whether they eventually reach $\top$. This set is also known as the *Rosolini dominance* [74] and denoted by Σ^0_1. When $\mathbb{S} = \Sigma^0_1$, open subsets $U : A \to \mathbb{S}$ can be understood as the semi-decidable subsets. Let $a \in A$ and let $s_0 \leq s_1 \leq \ldots$ be an increasing binary sequence representing $U(a)$. If $s_n = \top$ for some n, then $a \in U$, but we can never conclude $a \notin U$ by checking only a finite prefix of s. Under a realizability interpretation, s corresponds to a computation producing an infinite stream of digits which will eventually contain 1 if and only if $a \in U$. If furthermore A itself is enumerable, we obtain an enumeration of U. The Rosolini dominance is not well-behaved without countable choice. For example, it is not closed under enumerable disjunction. We circumvent this issue by using the initial σ-frame instead, which is closed under enumerable disjunction by definition.

An important requirement imposed on the set of open truth values is the *dominance axiom.* Consider inclusions of spaces $A \subseteq B \subseteq C$ such that A is open in B and B is open in C. In analytic topology, this implies that A is open in C. This is not automatic in synthetic topology, but holds if $S \subseteq \Omega$ is a *dominance* [74]:

Definition 2.6 (`Dominance.v:23`)**.** A subset $S \subseteq \Omega$ is a *dominance* if for all $p \in \Omega$ and $s \in S$ it holds that

$$(s \implies (p \in S)) \implies (s \wedge p) \in S. \tag{2.3}$$

Note that $p \in S$ and $(s \wedge p) \in S$ are themselves propositions, hence elements of Ω. Elements $s \in S$ are, via the inclusion $S \subseteq \Omega$, in particular propositions.

Rosolini [74] proved that Σ^0_1 is a dominance under the assumption of countable choice. It follows that $\mathbb{S}$ is a dominance if countable choice holds. But $\mathbb{S}$ being a dominance can be proved directly, and even without assuming countable choice:

Theorem 2.7 (`Dominance.v:32`)**.** *The Sierpinski space $\mathbb{S} \subseteq \Omega$ is a dominance.*

Proof. We prove the dominance property (2.3) for fixed $p \in \Omega$ using the induction principle of $\mathbb{S}$ as a free ω-cpo completion of $\mathbb{B}$. If $s = \top$ and $s \implies (p \in \mathbb{S})$, then in particular $p \in \mathbb{S}$ and thus $(s \wedge p) = p$ is in $\mathbb{S}$. If $s = \bot$, then $(s \wedge p) = \bot$, which is an element of $\mathbb{S}$. Now let $s = \bigvee_n s_n$ for an ascending chain $s_0 \leq s_1 \leq \ldots$ in $\mathbb{S}$. Suppose that $s \implies (p \in \mathbb{S})$ and that the dominance property (2.3) with s_n in place of s holds for all $n \in \mathbb{N}$. Combining this with $s_n \implies s$ and $s \implies p$ it follows that $s_n \wedge p$ is in $\mathbb{S}$ for all n. Thus

$$s \wedge p = (\bigvee_n s_n) \wedge p = \bigvee_n (s_n \wedge p)$$

by the distributive law, which is in $\mathbb{S}$. □

Given a dominance S and a set A, Rosolini constructs a *partial map classifier* of A, which is an object representing partial maps $B \rightharpoonup A$ whose domains of definition are open with respect to S. Following Escardó and Knapp [30], the partial map classifier can be defined as

$$\mathcal{L}_S A = \{(s, v) \mid s \in S, v : s \to A\}.$$

Here s is identified with the subsingleton set $\{* \mid s\}$. They refer to elements $(s, v) \in \mathcal{L}_S A$ as *partial elements.* v is the *value*, s its *extent.* Under a realizability interpretation and $S = \mathbb{S} = \Sigma^0_1$, maps $B \to \mathcal{L}_S A$ can be thought of as partial functions from B to A, in the sense that their interpretations yield potentially non-terminating computations producing results in A. The interpretation of constructive logic in the effective topos even validates the axiom that for every function $\mathbb{N} \to \mathcal{L}_S\mathbb{N}$ there merely exists a Turing machine which computes it [16, chapter 3].

If one uses the booleans $\{\bot, \top\}$ as set of open truth values S, then $\mathcal{L}_S A$ is the set of *decidably* partial elements. $\mathcal{L}_S A$ can then be described as the free partial order with bottom element over the discrete partial order A. Its underlying set is the sum $A + 1$, and all elements of A are greater than the element of 1. In this case it is thus decidable whether $x \in \mathcal{L}_S A$ represents a fully defined element (i.e. $x \in A$) or whether x is undefined (i.e. $x \in 1$), so that we may think of elements of $A + 1$ as *decidably* partial elements of A.

Altenkirch et al. [2] propose defining the partial map classifier of A as the QIIT $A_\bot$ described in Section 2.3. In our terminology, $A_\bot$ is the ω-cpo completion $(A + 1)_\omega$. Escardó and Knapp [30] mention that $A_\bot$ can be understood in terms of Rosolini's lifting construction. Indeed, $\mathcal{L}_\mathbb{S} A$ has the structure of an ω-cpo with bottom element under A: The structure map $e : A \to \mathcal{L}_\mathbb{S} A$ is defined by assigning to each element $a \in A$ the unique map $\top \to A$ with value a. For $v : s \to A$ and $v' : s' \to A$ in $\mathcal{L}_\mathbb{S} A$ let

$$(s, v) \leq (s', v') \iff ((s \implies s') \wedge v'_{|s} = v : s \to A.$$

This defines a partial order on $\mathcal{L}_\mathbb{S} A$. Its bottom element is the unique map $\bot \to A$. The join of an enumerable directed set $U = \{(s_u, v_u) \mid u \in U\} \subseteq \mathcal{L}_\mathbb{S} A$ is given by $(\bigvee_{u \in U} s_u, v)$, where v is defined by $v(x) = v_{u_0}(x)$ whenever $x \in \bigvee_{u \in U} s_u$ is in s_{u_0}. Thus there is a unique ω-continuous map $f : A_\bot \to \mathcal{L}_\mathbb{S} A$ which is compatible with the structure maps and preserves the bottom element. We can then show the following:

Proposition 2.8. *The map $f : A_\bot \to \mathcal{L}_\mathbb{S} A$ is an order isomorphism.*

Proof. First note that the projection $\mathcal{L}_\mathbb{S} A \to \mathbb{S}$ that sends a partial element (s, v) to its extent s is ω-continuous and preserves the bottom element. The unique map $A \to 1$ induces a map $A_\bot \to 1_\bot = \mathbb{S}$, which can equivalently be described as assigning to $x \in A_\bot$ the truth value

$$(\exists a \in A, \eta(a) = x) \in \Omega$$

by Proposition 2.5. (A direct proof of this can also be found in Gilbert [34].) Here the existential quantifier denotes *mere* existential quantification. By the universal property of $A_\bot$, the maps constructed so far commute with f, so if $f(x) = (s, v)$, then $s \iff \exists a \in A, x = \eta(a)$.

Now let us show that f exhibits $A_\bot$ as suborder of $\mathcal{L}_\mathbb{S} A$. Suppose $f(x) = (s, v)$ and $f(x') = (s', v')$ such that $(s, v) \leq (s', v')$ in $\mathcal{L}_\mathbb{S} A$. We show $x \leq x'$ by induction over x. If $x = \bot$, then trivially $x \leq x'$. If $x = \eta(a)$ for some $a \in A$, then $s' \geq s = \top$, hence $s' = \top$. From this it follows by our initial remark that there merely exists $a' \in A$ such that $x' = \eta(a')$. In particular, $a = v(*) = v'(*) = a'$, where $* \in \top$ is the unique element of the unit set, hence $x = x'$. Now let $x = \bigvee U$ be the join of a directed enumerable subset $U \subseteq A_\bot$. We may assume that for all $u \in U$, if $f(u) \leq f(x')$, then $u \leq x'$. Thus $u \leq x'$

because $f(u) \leq f(x) \leq f(x')$ for all u. But then $x = \bigvee U \leq x'$ by definition of least upper bound.

It remains to show that f is surjective and hence an order isomorphism. For this we must construct for each partial element $(s, v) \in \mathcal{L}_{\mathbb{S}}A$ an element $x \in A_\perp$ such that $f(x) = (s, v)$. We proceed by induction over s. We can set $x = \perp$ if $s = \perp$ and $x = \eta(v(*))$ if $s = \top$. Now let $s = \bigvee U$ be a directed enumerable join in $\mathcal{L}_{\mathbb{S}}A$. We may assume that for partial elements of the form $w : u \to A$ with $u \in U$ there merely exists $x \in A_\perp$ such that $f(x) = (u, w)$. Because $f : A_\perp \to \mathcal{L}_{\mathbb{S}}A$ was already proved to be the inclusion of a suborder,

$$V = \{x \in A_\perp \mid f(x) = (u, v_{|u}) \text{ for some } u \in U\}$$

embeds into U. By the induction hypothesis, it is isomorphic to U, hence directed and enumerable. Now $f(\bigvee V) = \bigvee f(V) = \bigvee_{u \in U}(u, v_{|u}) = (s, v)$. □

2.5 The lower reals

A *Dedekind cut* is a pair of sets of rational numbers (L, U) of the form $L = (\infty, x) \cap \mathbb{Q}$ and $U = (x, \infty) \cap \mathbb{Q}$ for some real number x. The condition that (L, U) is of this form can be stated purely in terms of rational numbers without referring to the real numbers, so the (Dedekind) real numbers $\mathbb{R}$ can be defined as the set of all pairs (L, U) satisfying these requirements; see e.g. Johnstone [50]. Constructively, even a bounded subset of $\mathbb{R}$ does not necessarily have a supremum. This is problematic in integration theory, because integrals of functions on non-compact spaces are constructed by approximating them from below.

A lower real is given only by the lower part L. Note that, constructively, U cannot be reconstructed from just L or vice-versa. In the setting of synthetic topology, it is natural to ask that the subsets L (and U) are valued in the Sierpinski space S, so that they correspond to subsets of $\mathbb{Q}$ which are open with respect to S. For Dedekind reals, this has been studied extensively by Lešnik [59]. The usage of the initial σ-frame $\mathbb{S}$ in the definition of Dedekind real numbers is also proposed in the HoTT book (Section 11.2) and has been formalized by Gilbert [34]. For us $\mathbb{S} = S$ is the Sierpinski space, so real numbers x given by open Dedekind cuts can be understood as those for which the predicates $q < x$ and $q > x$ on rational numbers q are semi-decidable. If x is a lower real, then only the predicate $q < x$ will be semi-decidable. We use the symbol $\mathbb{R}$ to refer to the Dedekind reals valued in $\mathbb{S}$ and likewise $\mathbb{R}_l$ for the set of lower reals valued in $\mathbb{S}$.

Definition 2.9 (`Rlow.v:46`). A *lower real* is an open subset $L : \mathbb{Q} \to \mathbb{S}$ of $\mathbb{Q}$ satisfying the following axioms:

- There merely exists $q \in \mathbb{Q}$ such that $L(q)$,

- for all $q \in \mathbb{Q}$, if $L(q)$ then there merely exists $q' > q$ such that $L(q')$, and
- for all $q < q' \in \mathbb{Q}$, if $L(q')$, then $L(q)$.

The set of all lower reals is denoted by $\mathbb{R}_l$. For $q \in \mathbb{Q}$ let

$$\underline{q} = \{p \in \mathbb{Q} \mid p < q\} \in \mathbb{R}_l.$$

The subset of *non-negative lower reals* is given by

$$\mathbb{R}_l^+ = \{L \in \mathbb{R}_l \mid \forall q \in \mathbb{Q}, q < 0 \implies L(q)\}.$$

Note that $\infty = \mathbb{Q} \in \mathbb{R}_l$ is a lower real, but that $-\infty$ (however it may be defined) is not in $\mathbb{R}_l$. In predicative foundations, the Dedekind or lower reals usually have to be parameterized by a universe level i, corresponding to the size of the set of truth values Ω_i the lower (and upper) cuts are valued in. The resulting set of reals will only be an element of the $(i+1)$th universe. Using the set of open truth values $\mathbb{S}$, we avoid this nuisance and obtain just one set of Dedekind and lower reals, respectively.

Crucial for the use of lower reals in integration theory is their order-theoretic structure:

Proposition 2.10 (`Rlow.v`)**.** *The lower reals endowed with the relation*

$$L_1 \leq L_2 \iff \forall q \in \mathbb{Q}, q \in L_1 \implies q \in L_2$$

for $L_1, L_2 \in \mathbb{R}_l$ are a partial order. Finite meets and enumerable joins in $\mathbb{R}_l$ exist, are computed pointwise and satisfy the distributivity law $x \wedge (\bigvee_{n\in\mathbb{N}} y_n) = \bigvee_{n\in\mathbb{N}} (x \wedge y)$. The suborder of non-negative lower reals $\mathbb{R}_l^+$ is a σ-frame. The map $q \mapsto \underline{q}$ exhibits $\mathbb{Q}$ as suborder of $\mathbb{R}_l$. $\square$

In view of Proposition 2.10, it is natural to wonder whether $\mathbb{R}_l$ is obtained by a completion process of $\mathbb{Q}$. This is indeed the case. Define a cover relation on $\mathbb{Q}$ by $q \lhd U$ for enumerable directed $U \subseteq \mathbb{Q}$ such that $\bigvee U$ exists and is equal to q. The embedding $\mathbb{Q} \subseteq \mathbb{R}_l$ preserves enumerable joins and thus induces an ω-continuous map $f : \mathbb{Q}_\omega \to \mathbb{R}_l$. Similarly we have $f^+ : (\mathbb{Q}^+)_\omega \to \mathbb{R}_l^+$, where $\mathbb{Q}^+$ is understood as an ω-cpo presentation with the restricted cover relation of $\mathbb{Q}$.

Theorem 2.11. *The unique ω-continuous maps $f : \mathbb{Q}_\omega \to \mathbb{R}_l$ and $f^+ : (\mathbb{Q}^+)_\omega \to \mathbb{R}_l^+$ under $\mathbb{Q}$ respectively $\mathbb{Q}^+$ are order isomorphisms.*

Noting that the two operations preserve covers, we conclude with Proposition 2.3 the following:

Corollary 2.12 (`Rlow.v`)**.** *Addition on $\mathbb{Q}$ and multiplication on $\mathbb{Q}^+$ extend uniquely to ω-continuous operations on $\mathbb{R}_l$ and $\mathbb{R}_l^+$, respectively.*

Our Coq formalization includes definitions of addition on $\mathbb{R}_l$ (`Rlow.v:330`) and multiplication of lower reals by rational numbers (`Rlow.v:1376`), but does not prove uniqueness as asserted by Corollary 2.12.

Multiplication cannot be (constructively) extended to an operation on all lower reals because it is not monotone. In terms of lower cuts, we have $q \in (L_1 + L_2)$ if and only if there merely exist $q_1 \in L_1$ and $q_2 \in L_2$ such that $q_1 + q_2 = q$, and similarly for the product $L_1 \cdot L_2$ if $L_1, L_2 \in \mathbb{R}_l^+$.

The statement analogous to Theorem 2.11 for the usual lower reals (which are not required to be valued in $\mathbb{S}$) and completion under arbitrary directed joins can be shown as follows. The proposed inverse g to f maps a lower real $L : \mathbb{Q} \to \Omega$ to the join $g(L) = \bigvee_{q \in L} \eta(q)$ in the completion of $\mathbb{Q}$ under arbitrary directed joins. This defines a continuous map which is compatible with the inclusions of $\mathbb{Q}$, hence $gf = \mathrm{id}$ by the universal property of the completion. On the other hand, $fg = \mathrm{id}$ because $L = \bigvee_{q \in \mathbb{Q}} \underline{q}$ for all L. Unfortunately, this proof does not directly transfer to our situation because lower reals $L : \mathbb{Q} \to \mathbb{S}$ are not necessarily enumerable in the sense that there is a surjection $\mathbb{N} \twoheadrightarrow L = \{q \in \mathbb{Q} \mid L(q)\}$, at least not in the absence of countable choice.

Proof of Theorem 2.11. For brevity, we only prove the statement about $\mathbb{R}_l$, the proof for $\mathbb{R}_l^+$ being similar. Note that the covers of $\mathbb{Q}$ are stable under binary joins, thus $\mathbb{Q}_\omega$ has binary joins and hence arbitrary enumerable joins. This allows us to construct a map $g : \mathbb{R}_l \to \mathbb{Q}_\omega$ as follows. Let $L \in \mathbb{R}_l$ and pick $q \in L$. For each $p \in \mathbb{Q}$, let $s \mapsto p_s$ be the unique ω-continuous map $\mathbb{S} \to \mathbb{Q}_\omega$ which sends $\perp$ to $\eta(q)$ and $\top$ to $\eta(p)$. Now set

$$g(L) = \bigvee_{p \in \mathbb{Q}} p_{L(p)}.$$

If $p \in L$, then $p_{L(p)} = \eta(p)$ by definition, and so $\bigvee_{q \in \mathbb{Q}} q_{L(q)} \geq \eta(p)$. Thus g is well-defined as it does not depend on the choice of q.

g is defined as composition of ω-continuous maps, so is ω-continuous itself. It is compatible with the structure maps $\mathbb{Q} \to \mathbb{R}_l$ and $\mathbb{Q} \to \mathbb{Q}_\omega$ because

$$g(\underline{q}) = \bigvee_{p \in \mathbb{Q}} p_{\underline{q}(p)} = \bigvee_{p < q} \eta(p) = \eta(q)$$

by definition of the cover relation on $\mathbb{Q}$. It follows that $gf = \mathrm{id}$ by the universal property of $\mathbb{Q}_\omega$.

Note that f preserves arbitrary enumerable joins (not necessarily directed) because the map $\mathbb{Q} \to \mathbb{R}_l$ preserves binary joins. Let $L \in \mathbb{R}_l$. It can be shown by induction over $L(p)$ that $f(p_{L(p)}) \leq L$ for all $p \in \mathbb{Q}$. Thus

$$f(g(L)) = f(\bigvee_{p \in \mathbb{Q}} p_{L(p)}) = \bigvee_{p \in \mathbb{Q}} f(p_{L(p)}) \leq L.$$

On the other hand, suppose $q \in L$ and let us show that $q \in f(g(L))$, i.e. that $L \leq f(g(L))$. Because L is a rounded lower subset of $\mathbb{Q}$, there merely exists $q' > q$ such that $q' \in L$. Then $f(q'_{L(q')}) = \underline{q'} \leq f(g(L))$, hence $q \in f(g(L))$. $\square$

2.6 Integrals and Valuations

In this section we define valuations, which play the role of measures but are defined only on opens, and integrals. We then prove a version of the Riesz theorem, which states that there is a one-to-one correspondence between valuations and integrals. Valuations are often preferred over measures in constructive mathematics because measures would have to be valued in the hyperreals [21]. They have a long tradition in the domain-theoretic semantics of probabilistic computations, see e.g. Jones and Plotkin [51]. It is observed there that, classically, valuations on compact Hausdorff spaces are in bijective correspondence with regular measures. Our proof of the Riesz theorem is inspired by Coquand and Spitters [22] and Vickers [91], who prove similar results in the setting of locales.

Fix a set A. Recall that $\mathcal{O}(A)$, the set of open subsets of A, is defined as the set of functions $A \to \mathbb{S}$. The σ-frame structures of $\mathbb{S}$ and $\mathbb{R}_l^+$ induce σ-frame structures on the sets of functions $\mathcal{O}(A)$ and $A \to \mathbb{R}_l^+$, with all structure defined pointwise.

Definition 2.13 (`Valuations.v:49`). An (ω-continuous) *valuation* on a set A is an ω-continuous map $\mu : \mathcal{O}(A) \to \mathbb{R}_l^+$ preserving the bottom element that satisfies the *modularity law*

$$\mu(U) + \mu(V) = \mu(U \cup V) + \mu(U \cap V).$$

for all opens $U, V \in \mathcal{O}(A)$. μ is a *sub-probability* valuation if $\mu(A) \leq 1$. The set of all valuations on A is denoted by $\mathfrak{V}(A)$ and the set of sub-probability valuations by $\mathfrak{V}_{\leq 1}(A)$.

Let $r : \mathbb{S} \to \mathbb{R}_l$ be the unique ω-continuous map such that $r(\bot) = 0$ and $r(\top) = 1$. By postcomposition we obtain a map $(A \to \mathbb{S}) \to (A \to \mathbb{R}_l)$ that assigns to each open $U \in \mathcal{O}(A) = (A \to \mathbb{S})$ its *(real) indicator function* $\mathbb{1}_U = r(U) : A \to \mathbb{R}_l$.

The map $U \mapsto \mathbb{1}_U$ is an order embedding, and so we can equivalently think of a valuation μ as assigning lower reals to a certain subset of functions $A \to \mathbb{R}_l^+$. The Riesz theorem states that every valuation μ can be extended to a lower integral, which is a function defined on *all* maps $A \to \mathbb{R}_l^+$, and that every lower integral is determined by its restriction to indicator functions.

Definition 2.14 (`LowerIntegrals.v:76`). A *lower integral* on A is an ω-continuous map $\mathcal{I} : (A \to \mathbb{R}_l^+) \to \mathbb{R}_l^+$ preserving the bottom element that is furthermore additive, i.e. satisfies

$$\mathcal{I}(f+g) = \mathcal{I}(f) + \mathcal{I}(g)$$

for all $f, g : A \to \mathbb{R}_l^+$. $\mathcal{I}$ is a *sub-probability* lower integral if $\mathcal{I}(\mathbb{1}_A) \leq 1$. The set of all lower integrals on A is denoted by $\mathfrak{G}(A)$ and the set of sub-probability lower integrals by $\mathfrak{G}_{\leq 1}(A)$.

The reader might wonder at this point why we need the generality of *sub*-probability valuations and integrals, as opposed to probability valuations and integrals, which would assign to (the indicator function of) the whole space the value 1. Valuations and integrals on some set A form partial orders, with ordering defined pointwise. Now, if we restrict to proper probability valuations and integrals, these orders will usually not have least elements (consider, for example, valuations on the set of two elements). On the other hand, for their sub-probabilistic versions we have the following, which will be crucial for the interpretation of fixpoint operators in Section 2.9:

Proposition 2.15. *The inclusions $\mathfrak{V}_{\leq 1}(A) \subseteq \mathfrak{V}(A) \subseteq (\mathcal{O}(A) \to \mathbb{R}_l^+)$ and $\mathfrak{G}_{\leq 1}(A) \subseteq \mathfrak{G}(A) \subseteq ((A \to \mathbb{R}_l^+) \to \mathbb{R}_l^+)$ are embeddings of ω-cpos with bottom elements.* □

Proposition 2.16. *Every lower integral $\mathcal{I}$ is compatible with multiplication by scalars from $\mathbb{R}_l^+$, in the sense that $\mathcal{I}(af) = a\mathcal{I}(f)$ for all $a \in \mathbb{R}_l^+$ and $f : A \to \mathbb{R}_l^+$. In particular, lower integrals are linear over $\mathbb{R}_l^+$.*

Proof. If $a \in \mathbb{N}$, then $\mathcal{I}(af) = \mathcal{I}(f + \cdots + f) = a\mathcal{I}(f)$ because $\mathcal{I}$ is additive. Thus if $a = \frac{m}{n}$ is a positive rational, then $n\mathcal{I}(af) = \mathcal{I}(naf) = m\mathcal{I}(f)$, hence $\mathcal{I}(af) = \frac{m}{n}\mathcal{I}(f)$. If U is a directed enumerable set of lower reals such that for each $a \in U$ we have $\mathcal{I}(af) = a\mathcal{I}(f)$ for all f, then

$$\mathcal{I}((\bigvee U)f) = \mathcal{I}(\bigvee_{a \in U}(af)) = (\bigvee U)\mathcal{I}(f)$$

by ω-continuity of $\mathcal{I}$ and multiplication, so $\mathcal{I}$ is compatible with multiplication by $\bigvee U$. Because $\mathbb{R}_l^+$ is the ω-cpo completion of $\mathbb{Q}^+$ (Theorem 2.11), it follows that $\mathcal{I}$ is compatible with scalar multiplication by arbitrary non-negative lower reals a. □

We are now ready to state the central result of this section.

Theorem 2.17 (Riesz)**.** *The assignment*

$$\mathcal{I} \mapsto (U \mapsto \mathcal{I}(\mathbb{1}_U))$$

defines a map $\mathfrak{G}(A) \to \mathfrak{V}(A)$ that restricts to a map $\mathfrak{G}_{\leq 1}(A) \to \mathfrak{V}_{\leq 1}(A)$. Both maps are order isomorphisms.

We begin the proof by showing that restrictions of lower integrals to indicator functions are valuations.

Lemma 2.18 (`Riesz1.v:22`). *Let $\mathcal{I}$ be an integral on A. Then $\mu_{\mathcal{I}} : U \mapsto \mathcal{I}(\mathbb{1}_U)$ is a valuation on A. If $\mathcal{I}$ is a sub-probability integral, then $\mu_{\mathcal{I}}$ is a sub-probability valuation.*

Proof. Recall that $\mathbb{1}_U$ is obtained by postcomposing $U : A \to \mathbb{S}$ with the unique ω-continuous map $r : \mathbb{S} \to \mathbb{R}_l^+$ that satisfies $r(\bot) = 0$ and $r(\top) = 1$. Thus $U \mapsto \mathbb{1}_U$ is ω-continuous, too, hence ω-continuity of $\mu_{\mathcal{I}}$ follows from ω-continuity of $\mathcal{I}$. By definition $\mu_{\mathcal{I}}(A) = \mathcal{I}(\mathbb{1}_A)$, so if the latter is ≤ 1, then so is the former.

What remains to be shown is that $\mu_{\mathcal{I}}$ satisfies the modularity law, i.e. that

$$\mathcal{I}(\mathbb{1}_{U \cup V}) + \mathcal{I}(\mathbb{1}_{U \cap V}) = \mathcal{I}(\mathbb{1}_U) + \mathcal{I}(\mathbb{1}_V).$$

holds for all $U, V \in \mathcal{O}(A)$. By linearity of $\mathcal{I}$ and the definition of indicator functions, it will suffice to show that for all $s, t \in \mathbb{S}$ it holds that

$$r(s \vee t) + r(s \wedge t) = r(s) + r(t), \tag{2.4}$$

and we will do so by induction over s. If $s = \top$, both sides are equal to $1 + r(t)$, and if $s = \bot$, then both sides are equal to $r(t)$. Now let $s = \bigvee U$ for an enumerable directed subset $U \subseteq \mathbb{S}$, and suppose that equation (2.4) holds with u in place of s for all $u \in U$. Using the fact that the involved operations binary meet and join with t, addition and r are all ω-continuous, we compute

$$\begin{aligned} r(s \vee t) + r(s \wedge t) &= \bigvee_{u \in U} (r(u \vee t) + r(u \wedge t)) \\ &= \bigvee_{u \in U} (r(u) + r(t)) \\ &= r(s) + r(t). \end{aligned}$$

□

Next we construct the extension $\int - \, d\mu$ of a valuation μ to an integral. Fix μ.

Definition 2.19. Let $f : A \to \mathbb{R}_l^+$. The lower μ*-integral* $\int f \, d\mu \in \mathbb{R}_l^+$ is defined as follows. For $q \in \mathbb{Q}_+$ let

$$[f > q] = \{x \in A \mid q < f(x)\};$$

it is an open subset of A. Let

$$s_{f,m,n} = \sum_{i=1}^{mn} \frac{1}{m} \mu([f > \frac{i}{m}])$$

for $m, n \in \mathbb{N}$. Now

$$\int f d\mu = \bigvee_{m,n \in \mathbb{N}} s_{f,m,n}.$$

The main difficulty in showing that $f \mapsto \int f \, d\mu$ is indeed a lower integral is the verification of linearity. Our main tool will be the *generalized modularity lemma*, originally due to Horn and Tarski [42, corollary 1.3] in the special case of boolean algebras. More recent references are Coquand and Spitters [22] and Vickers [91]; the latter also contains a proof of the version that will be used here. Generalized modularity is phrased in terms of the following construction, which in the special case $L = \mathcal{O}(A)$ can be understood as the submonoid of functions $A \to \mathbb{R}_l^+$ generated by the indicator functions $\mathbb{1}_U$ for $U \in \mathcal{O}(A)$.

Definition 2.20. Let L be a distributive lattice with bottom element. The *modular monoid* $M(L)$ is the commutative monoid (written additively) generated by the carrier of L subject to

$$a + b = (a \wedge b) + (a \vee b)$$

for all $a, b \in L$, and $0 = \bot$.

Note that the modularity law and the preservation of bottom elements guarantee precisely that valuations $\mu : \mathcal{O}(A) \to \mathbb{R}_l$ factor uniquely as monoid homomorphism $L(\mathcal{O}(A)) \to \mathbb{R}_l$.

Lemma 2.21 (Generalized Modularity Lemma)**.** *Let L be a distributive lattice and $x_1, \ldots, x_n \in L$. Then in $M(L)$ we have*

$$\sum_{i=1}^{n} x_i = \sum_{k=1}^{n} \bigvee \{x_I \mid I \subseteq \{1, \ldots, n\}, |I| = k\}$$

where $x_I = \bigwedge \{x_i \mid i \in I\}$ for decidable $I \subseteq \{1, \ldots, n\}$.

Let $q \in \mathbb{Q}^+$ and $f : A \to \mathbb{R}_l^+$. Define $[f > q]_0 \subseteq A$ to be $[f > q]$ if $q > 0$ and equal to A if $q = 0$.

Lemma 2.22. *Let $f, g : A \to \mathbb{R}_l^+$. Then in $M(\mathcal{O}(A))$ we have*

$$\begin{aligned} &\sum_{k=1}^{n} ([f > k] + [g > k]) \\ = &\sum_{k=1}^{2n} \bigvee \{[f > i]_0 \wedge [g > j]_0 \mid i, j \in \{0, \ldots, n\}, i + j = k\} \end{aligned}$$

for all $n > 0$.

Proof. Regarding the left-hand side as a sum with $2n$ summands, we have by the Generalized Modularity Lemma 2.21

$$\begin{aligned}&\sum_{k=1}^{n}([f>k]+[g>k])\\ =&\sum_{k=1}^{2n}\bigvee\{[f>I]\wedge[g>J]\mid I,J\subseteq\{1,\dots,n\},|I|+|J|=k\}\end{aligned}$$

where $[f>I]=\bigwedge\{[f>i]\mid i\in I\}$ and similarly $[g>J]=\bigwedge\{[g>j]\mid j\in J\}$. Because $[f>i_0]\supseteq[f>i_1]$ whenever $i_0\leq i_1$, we have $[f>I]=[f>\bigvee I]$ for inhabited $I\subseteq\{1,\dots,n\}$. If I is empty, then $\bigvee I=0$ and hence $[f>I]=A=[f>\bigvee I]_0$. It follows that $[f>I]\leq[f>\{1,\dots,\ell\}]_0=[f>\ell]_0$ if $I\subseteq\{1,\dots,n\}$ has ℓ elements and similarly for $[g>J]$. Discarding small elements from joins, we obtain

$$\begin{aligned}&\bigvee\{[f>I]\wedge[g>J]\mid I,J\subseteq\{1,\dots,n\},|I|+|J|=k\}\\ =&\bigvee\{[f>i]_0\wedge[g>j]_0\mid i,j\in\{0,\dots,n\},i+j=k\}\end{aligned}$$

for $1\leq k\leq 2n$. □

Lemma 2.23. *Let $f:A\to\mathbb{R}_l^+$. Suppose m,m',n,n' are positive integers such that $n\leq n'$ and $m\mid m'$ (i.e. m divides m'). Then $s_{f,m,n}\leq s_{f,m',n'}$. The family $(s_{f,m,n})_{m,n}$ is directed.*

Proof. The inequality is clear if $m=m'$, so by transitivity it will suffice to prove the inequality for $n'=n$ and $m'=mq$ for some integer $q>0$. Dividing i by q with remainder, we obtain for each integer $1\leq i\leq m'n$ unique integers $0\leq k\leq mn-1$ and $1\leq j\leq q$ such that $i=qk+j$. Thus

$$s_{f,m',n}=\frac{1}{m'}\sum_{i=1}^{m'n}\mu([f>\frac{i}{m'}])=\frac{1}{m}\sum_{k=0}^{mn-1}\frac{1}{q}\sum_{j=1}^{q}\mu([f>\frac{qk+j}{m'}]).$$

Now $[f>\frac{qk+j}{m'}]\geq[f>\frac{q(k+1)}{m'}]=[f>\frac{k+1}{m}]$, hence

$$\frac{1}{m}\sum_{k=0}^{mn-1}\frac{1}{q}\sum_{j=1}^{q}\mu([f>\frac{qk+j}{m'}])\geq\frac{1}{m}\sum_{k=0}^{mn-1}\mu([f>\frac{k+1}{m}])=s_{f,m,n}$$

by monotonicity of μ. Both $m\mid m'$ and $n\leq n'$ are directed partial orders on the positive integers, thus $(s_{f,m,n})_{m,n}$ is a directed family. □

Lemma 2.24. *Let μ be a valuation on A. Then the assignment $f\mapsto\int f\,d\mu$ is a lower integral.*

Proof. We verify the conditions of Definition 2.14.

Preservation of $\perp$. If $f = 0$ is the constant function with value zero, then $[f > q] = \emptyset$ for all $q > 0$, hence $s_{f,m,n} = 0$ for all m, n, so $\int f\, d\mu = \bigvee_{m,n} s_{f,m,n} = 0$.

ω-continuity. The integral is defined in terms of the following operations, all of which are ω-continuous: $f \mapsto [f > q]$, μ, addition, scalar multiplication and join.

Additivity. Let $f, g : A \to \mathbb{R}_l^+$. Let $m, n \geq 1$. Note that $\int f\, d\mu + \int g\, d\mu = \bigvee_{m,n} (s_{f,m,n} + s_{g,m,n})$ because the families $(s_{-,m,n})_{mn}$ are directed (Lemma 2.23) and addition is ω-continuous. Application of Lemma 2.22 for the functions mf and mg gives

$$s_{f,m,n}+s_{g,m,n} = \frac{1}{m}\sum_{k=1}^{2mn} \mu(\underbrace{\bigcup\{[f > \frac{i}{m}]_0 \cap [g > \frac{j}{m}]_0 \mid 0 \leq i, j \leq mn, i + j = k\}}_{\subseteq [f+g>\frac{k}{m}]}),$$

which is $\leq s_{(f+g),m,2n}$. Letting n and m vary, we conclude $\int f\, d\mu + \int g\, d\mu \leq \int (f + g)\, d\mu$.

On the other hand, let $q \in \mathbb{Q}$ such that $q < \int f + g\, d\mu$. We will show $q < \int f\, d\mu + \int g\, d\mu$. By definition of $\int - \, d\mu$ as a join, there merely exist $m, n \in \mathbb{N}$ such that $q < s_{f+g,m,n}$. Thus there are rational numbers $q_k < \mu([f + g > \frac{k}{m}])$ for $1 \leq k \leq nm$ such that $q = \frac{1}{m}\sum_{k=1}^{mn} q_k$. We have

$$[f + g > \frac{k}{m}] = \bigcup_{m|m'} \bigcup\{[f > \frac{i}{m'}] \cap [g > \frac{j}{m'}] \mid i, j \in \mathbb{N}, \frac{i + j}{m'} = \frac{k}{m}\}$$

for each k and the outer union on the right-hand side is directed, with upper bounds given by common multiples of the m'. Thus μ commutes with the outer union.

It follows that for each k there is m'_k such that

$$q_k < \mu(\bigcup\{[f > \frac{i}{m'_k}] \cap [g > \frac{j}{m'_k}] \mid i, j \in \mathbb{N}, \frac{i + j}{m'_k} = \frac{k}{m}\}). \tag{2.5}$$

By taking upper bounds wrt. divisibility, we may assume $m'_k = m'$ for all k

and a single m' such that $m|m'$. We obtain

$$
\begin{aligned}
&\int f\,d\mu + \int f\,d\mu \\
&\geq s_{f,m',n} + s_{g,m',n} \\
&= \frac{1}{m'} \sum_{\ell=1}^{2m'n} \mu\Big(\bigcup \{[f > \frac{i}{m'}] \cap [g > \frac{j}{m'}] \mid i,j \in \mathbb{N}, i+j=\ell\}\Big) \\
&= \frac{1}{m} \sum_{\ell'=0}^{2mn-1} \frac{m}{m'} \sum_{\ell''=1}^{m'/m} \mu\Big(\bigcup \{[f > \frac{i}{m'}] \cap [g > \frac{j}{m'}] \mid i,j \in \mathbb{N}, i+j=\ell' \frac{m'}{m} + \ell''\}\Big)
\end{aligned}
$$

by decomposing the index ℓ as $\ell = \ell' \frac{m}{m'} + \ell''$. Now

$$
\begin{aligned}
&\bigcup \{[f > \frac{i}{m'}] \cap [g > \frac{j}{m'}] \mid i,j \in \mathbb{N}, i+j=\ell' \frac{m'}{m} + \ell''\} \\
\supseteq &\bigcup \{[f > \frac{i}{m}] \cap [g > \frac{j}{m}] \mid i,j \in \mathbb{N}, i+j=\ell'+1\},
\end{aligned}
$$

for all ℓ and ℓ', which is independent of ℓ'. Thus

$$
\begin{aligned}
&\int f\,d\mu + \int f\,d\mu \\
&\geq \frac{1}{m} \sum_{k=1}^{2mn} \underbrace{\mu\Big(\bigcup \{[f > \frac{i}{m}] \cap [g > \frac{j}{m}] \mid i,j \in \mathbb{N}, i+j=k\}\Big)}_{>q_k \text{ if } k \leq nm} \\
&> \frac{1}{m} \sum_{k=1}^{mn} q_k \\
&= q
\end{aligned}
$$

where we reindexed with $k = \ell' + 1$ and used equation (2.5). $q < \int f + g\,d\mu$ was arbitrary, hence $\int f + g\,d\mu \leq \int f\,d\mu + \int g\,d\mu$. □

Proof of Theorem 2.17. By Lemma 2.18, the restriction $\mu_{\mathcal{I}}$ of an integral $\mathcal{I}$ to indicator functions is a valuation, and by Lemma 2.24 the assignment $f \mapsto \int f\,d\mu$ is an integral for all valuations μ. The two functions are monotone and restrict to functions on sub-probability valuations and integrals. It remains to show that

(1) $\int -\,d\mu$ is an extension of μ, i.e. $\int \mathbb{1}_U\,d\mu = \mu(U)$ for all opens $U \in \mathcal{O}(A)$, and

(2) every integral is uniquely determined by its value on indicator functions.

(1). Let $U \in \mathcal{O}(A)$ be an open subset. Then $[\mathbb{1}_U > q] = \emptyset$ for all $q \geq 1$, and $[\mathbb{1}_U > q] = U$ for all $q < 1$. Thus

$$s_{\mathbb{1}_U,m,n} = \frac{1}{m}\sum_{i=1}^{mn} \mu([\mathbb{1}_U > \frac{i}{m}]) = \frac{1}{m}\sum_{i=1}^{m-1} \mu(U) = \frac{m-1}{m}\mu(U)$$

for all $m, n > 1$, and we conclude $\int \mathbb{1}_U \, d\mu = \bigvee_{m>0} \frac{m-1}{m}\mu(U) = \mu(U)$.

(2). Let $\mathcal{I}$ be an integral and let $f : A \to \mathbb{R}_l^+$. Then

$$f = \bigvee_{m,n \geq 1} \frac{1}{m} \sum_{i=1}^{mn} \mathbb{1}_{[f > \frac{i}{m}]}.$$

and this join is directed (for the same reason that $(s_{f,m,n})_{mn}$ is a directed family). By linearity (Proposition 2.16) and ω-continuity of $\mathcal{I}$, we have

$$\mathcal{I}(f) = \bigvee_{m,n \geq 1} \frac{1}{m} \sum_{i=1}^{mn} \mathcal{I}(\mathbb{1}_{[f > \frac{i}{m}]}),$$

thus $\mathcal{I}$ is uniquely determined by its restriction to indicator functions. □

2.7 The Giry monad

By definition, there are inclusions $\mathfrak{G}_{\leq 1}(A) \subseteq \mathfrak{G}(A) \subseteq ((A \to \mathbb{R}_l^+) \to \mathbb{R}_l^+)$. The operator $\mathrm{Cont}_{\mathbb{R}_l^+} : A \mapsto ((A \to \mathbb{R}_l^+) \to \mathbb{R}_l^+)$ is the *continuation monad* [65] instantiated with $\mathbb{R}_l^+$. As we are working internally (i.e. an internal monad corresponds to an external *strong* monad), monad structure on an operator $M : \mathrm{Set} \to \mathrm{Set}$ is given by *unit* maps $\eta : A \to M(A)$ and *bind* maps $\gg\!=: M(A) \times (A \to M(B)) \to M(B)$ for all sets A, B, which satisfy unit and associativity laws. In case of the continuation monad $M = \mathrm{Cont}_{\mathbb{R}_l^+}$,

$$\eta(a) = (f \mapsto f(a))$$

is the map that evaluates a given $f : A \to \mathbb{R}_l^+$ at a certain $a \in A$, and bind is given by

$$x \gg\!= y = (f \mapsto x(a \mapsto y(a)(f))),$$

where $x \in M(A)$, $y : A \to M(B)$, $f : A \to \mathbb{R}_l^+$ and $a \in A$.

By the Riesz Theorem 2.17, $\mathfrak{G}(A) \cong \mathfrak{V}(A)$ and $\mathfrak{G}_{\leq 1}(A) \cong \mathfrak{V}_{\leq 1}(A)$. This justifies defining the Giry monad of (sub-probability) valuations as follows:

Definition and Proposition 2.25 (`Giry.v`). *The unit and bind operations of the continuation monad* $\mathrm{Cont}_{\mathbb{R}_l^+}$ *restrict to operations on (sub-probability) integrals. The* (sub-probabilistic) Giry monad *is given by the operator* $A \mapsto \mathfrak{G}(A)$ *(resp.* $A \mapsto \mathfrak{G}_{\leq 1}(A)$*) and the restricted unit and bind operations of the continuation monad.*

Proof. We need to show stability of $\mathfrak{G}$ and $\mathfrak{G}_{\leq 1}$ under η and $\gg\!=$. The verification of the rules of lower integrals is done by unfolding the pointwise definition of addition and the partial ordering on functions $A \to \mathbb{R}_l^+$. We show how some of the rules can be derived, the other proofs being similar.

If $a \in A$ and $f, g : A \to \mathbb{R}_l^+$, then $\eta(a)(f+g) = (f+g)(a) = f(a) + g(a) = \eta(a)(f) + \eta(a)(g)$, thus $\eta(a)$ is linear. Let $\mathcal{I} \in \mathfrak{G}(A)$ and $\mathcal{J} : A \to \mathfrak{G}(B)$. ω-continuity of $\mathcal{I} \gg\!= \mathcal{J}$ can be seen as follows. Let $U \subseteq (B \to \mathbb{R}_l^+)$ be a directed enumerable subset of the function space. Then for each $a \in A$ it holds that $\mathcal{J}(a)(\bigvee U) = \bigvee_{f \in U} \mathcal{J}(a)(f)$ because $\mathcal{J}(a)$ is ω-continuous. Thus

$$
\begin{aligned}
(\mathcal{I} \gg\!= \mathcal{J})(\bigvee U) &= \mathcal{I}(a \mapsto \bigvee_{f \in U} \mathcal{J}(a)(f)) \\
&= \mathcal{I}(\bigvee_{f \in U} (a \mapsto \mathcal{J}(a)(f))) \\
&= \bigvee_{f \in U} \mathcal{I}(a \mapsto \mathcal{J}(a)(f)) \\
&= \bigvee_{f \in U} (\mathcal{I} \gg\!= \mathcal{J})(f)
\end{aligned}
$$

using the pointwise definition of joins on $A \to \mathbb{R}_l^+$ and the ω-continuity of $\mathcal{I}$.

We have $\eta(a)(\mathbb{1}_A) = \mathbb{1}_A(a) = 1$ for all $a \in A$, so η is valued in sub-probability integrals. If $\mathcal{I} \in \mathfrak{G}_{\leq 1}(A)$ and $\mathcal{J} : A \to \mathfrak{G}_{\leq 1}(B)$, then $a \mapsto \mathcal{J}(a)(\mathbb{1}_B)$ is a function $\leq \mathbb{1}_A$ because $\mathcal{J}(a)$ is a sub-probability integral on B for all a. Thus $(\mathcal{I} \gg\!= \mathcal{J})(\mathbb{1}_B) \leq \mathcal{I}(\mathbb{1}_A) \leq 1$ by monotonicity of $\mathcal{I}$ and $\mathcal{I}$ being sub-probabilistic. □

Vickers [91] proves that the variant of the Giry monad on the category of locales is commutative. Commutativity of $\mathfrak{G}$ would mean that for $\mathcal{I} \in \mathfrak{G}(A)$ and $\mathcal{J} \in \mathfrak{G}(B)$ the two integrals

$$
(\mathcal{I} \blacktriangleright \mathcal{J})(f) = \mathcal{I}(a \mapsto \mathcal{J}(b \mapsto f(a,b))) \tag{2.6}
$$

and

$$
(\mathcal{I} \blacktriangleleft \mathcal{J})(f) = \mathcal{J}(b \mapsto \mathcal{I}(a \mapsto f(a,b)))
$$

on $A \times B$ agree.[2] In classical mathematics, this corresponds to the Fubini theorem

$$
\int_A \left(\int_B f(a,b)\, db \right) da = \int_B \left(\int_A f(a,b)\, da \right) db = \int_{A \times B} f(a,b)\, d(a,b)
$$

and uniqueness of product measures.

[2]Vickers uses $\lhd$ and $\rhd$ instead of $\blacktriangleleft$ and $\blacktriangleright$ here; we reserve unfilled triangles for the (unrelated) cover relations of ω-cpo presentations.

However, the proof given in Vickers [91] does not directly translate to our setting because it relies on the product $X \times Y$ of locales being dual to the coproduct $\mathcal{O}(X) \otimes \mathcal{O}(Y)$ of underlying frames. In synthetic topology, this corresponds to products having the product topology:

Definition 2.26. Let A and B be sets. For $U \in \mathcal{O}(A)$ and $V \in \mathcal{O}(B)$, let

$$U \times V = \{(a, b) \in A \times B \mid a \in U \wedge b \in V\} \in \mathcal{O}(A \times B);$$

it is open because $\mathbb{S}$ is closed under binary meets. $A \times B$ has the *product topology* if $\mathcal{O}(A \times B) \subseteq \mathcal{P}(A \times B)$ is the least subset that is closed under enumerable joins and contains the sets $U \times V$ for all $U \subseteq A$ and $V \subseteq B$ open.

Note that we require that the topology on $A \times B$ is generated by the basic opens $U \times V$ under enumerable unions, as opposed to arbitrary ones. Our notion of product topology is in a sense weaker than the one that can be found in Lešnik [59, definitions 2.57 and 2.55]. There it is required that every open is an overt (e.g. countable in our case) union of the basic opens $U \times V$, while for us the opens need only be *generated* by basic opens under enumerable unions. The situation is comparable to the initial σ-frame and the Rosolini dominance: In the presence of countable choice, the two definitions are equivalent.

The problem with the Fubini theorem in synthetic topology is that $A \times B$ does not always have the product topology. Fortunately, $A \times B$ does have the product topology in many special cases. Lešnik proves that if A and B are *strongly locally compact*, then $A \times B$ has the product topology ([59], proposition 2.59). Thus finite products of countable discrete spaces and locally compact metric spaces (e.g. $\mathbb{R}$ under suitable hypotheses, see Section 2.8) behave well, and our Fubini theorem applies.

Theorem 2.27 (Fubini). *Let $\mathcal{I} \in \mathfrak{G}(A)$ and $\mathcal{J} \in \mathfrak{G}(B)$ for sets A, B. Suppose that $A \times B$ has the product topology. Then the two integrals $\mathcal{I} \blacktriangleleft \mathcal{J}$ and $\mathcal{I} \blacktriangleright \mathcal{J}$ on $A \times B$ agree.*

The proof of Theorem 2.27 will occupy the remainder of Section 2.7. It is a direct translation of the proof given by Vickers [91] for locales.

Theorem 2.28 (Principle of inclusion and exclusion, [91]). *Let L be a lattice with bottom element. Then for all $x_1, \ldots, x_n \in L$ it holds in $M(L)$ that*

$$(\bigvee_{i=1}^{n} x_i) + \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ |I| \text{ is even}}} x_I = \sum_{\substack{|I| \subseteq \{1,\ldots,n\} \\ |I| \text{ is odd}}} x_I$$

where $x_I = \bigwedge_{i \in I} x_i$ for $I \subseteq \{1, \ldots, n\}$ decidable.

Lemma 2.29 ([91]). *Let $u_1, \ldots, u_n \in \mathbb{R}_l$ and v be lower reals. Then the equation $\sum_{i=1}^{n} u_i + x = v$ has at most one solution x such that $u_i \leq x$ for all i.*

Note that Lemma 2.29, as stated in the reference refers to the standard lower reals, which are not required to be valued in $\mathbb{S}$. However, the proof given there also works for our lower reals; moreover, the open lower reals embed into the standard lower reals so that uniqueness for the latter implies uniqueness for the former.

Lemma 2.30. *Let $U \in \mathcal{O}(A)$ and $V \in \mathcal{O}(B)$ be opens in sets A, B. Then for all $a \in A$ and $b \in B$ it holds that*

$$\mathbb{1}_{U\times V}(a,b) = \mathbb{1}_U(a) \cdot \mathbb{1}_V(b).$$

Proof. By definition of indicator functions and $U \times V$, the lemma will follow if we can show

$$r(s \wedge t) = r(s)r(t)$$

for all $s, t \in \mathbb{S}$, where $r : \mathbb{S} \to \mathbb{R}_l$ is the unique map of ω-cpos satisfying $r(\bot) = 0$ and $r(\top) = 1$. For fixed t, the two ω-continuous maps $s \mapsto r(s \wedge t)$ and $s \mapsto r(s)r(t)$ agree for $s = \bot$ and $s = \top$, so they agree by the universal property of $\mathbb{S} = \mathbb{B}_\bot$. □

Proof of Theorem 2.27. For $U \in \mathcal{O}(A)$ and $V \in \mathcal{O}(B)$, we compute with Lemma 2.30

$$\begin{aligned}(\mathcal{I} \blacktriangleleft \mathcal{J})(\mathbb{1}_{U\times V}) &= \mathcal{J}(b \mapsto \mathcal{I}(a \mapsto \mathbb{1}_U(a)\mathbb{1}_V(b))) \\ &= \mathcal{J}(\mathcal{I}(\mathbb{1}_U)\mathbb{1}_V) \\ &= \mathcal{I}(\mathbb{1}_U)\mathcal{J}(\mathbb{1}_V)\end{aligned}$$

and hence by symmetry

$$(\mathcal{I} \blacktriangleleft \mathcal{J})(\mathbb{1}_{U\times V}) = (\mathcal{I} \blacktriangleright \mathcal{J})(\mathbb{1}_{U\times V}) = \mathcal{I}(\mathbb{1}_U)\mathcal{J}(\mathbb{1}_V).$$

Because integrals are uniquely determined by their restriction to valuations, it will be sufficient to show that a valuation μ on $A \times B$ is in turn uniquely determined by its restriction to opens of the form $U \times V$. $A \times B$ has the product topology, so $\mathcal{O}(A \times B)$ is the least set containing subsets of the form $U \times V$ with $U \subseteq A$ and $V \subseteq B$ open that is closed under enumerable unions. Equivalently, $\mathcal{O}(A \times B)$ is generated under *directed* enumerable unions from opens of the form $U_1 \times V_1 \cup \cdots \cup U_n \times V_n$ for $U_i \subseteq A$ open and $V_i \subseteq B$ open, $1 \le i \le n$. It will thus suffice to prove that μ is uniquely determined by its value on finite unions of products of opens. Applying the principle of inclusion and exclusion (Theorem 2.28), we obtain

$$\mu\Big(\bigcup_{i=1}^{n}(U_i \times V_i)\Big) + \sum_{\substack{I\subseteq\{1,\ldots,n\} \\ |I| \text{ is even}}} \mu(U_I \times V_I) = \sum_{\substack{|I|\subseteq\{1,\ldots,n\} \\ |I| \text{ is odd}}} \mu(U_I \times V_I),$$

where $U_I = \bigcap_{i\in I} U_i$ and $V_I = \bigcap_{i\in I} V_i$ (hence $\bigcap_{i\in I}(U_i \times V_i) = U_I \times V_I$). By monotonicity of μ, it holds that $\mu(U_I \times V_I) \le \mu(\bigcup_{i=1}^{n}(U_i \times V_i))$, so by Lemma 2.29 the values $\mu(U_I \times V_I)$ uniquely determine $\mu(\bigcup_{i=1}^{n}(U_i \times V_i))$. □

2.8 The Lebesgue valuation

Having studied valuations in general, we now turn to constructing a concrete valuation on a non-discrete space: The Lebesgue valuation on the reals. For this we will need that the intrinsic topology of the Dedekind reals agrees with the topology that is induced by the Euclidean metric, i.e. that $\mathbb{R}$ is *metrizable* [59]. We proceed by defining a σ-frame of formal real opens and state metrizability as an isomorphism between the formal and the intrinsic real opens. The Lebesgue valuation can then be defined by a universal property.

Definition 2.31. The partial order L is the least suborder of $\mathcal{P}(\mathbb{Q})$ containing the sets $(a, b) = \{x \in \mathbb{Q} \mid a < q < b\}$ for all $a, b \in \mathbb{Q}$ that is closed under binary unions.

Every element $x \in L$ has a unique presentation as a disjoint union

$$x = (a_1, b_1)\,\dot{\cup} \ldots \dot{\cup}(a_n, b_n)$$

for rational numbers a_i, b_i such that $a_i < b_i \leq a_{i+1}$ for $i = 1, \ldots, n-1$. We refer to the elements (a_i, b_i) as the *connected components* of x. The decomposition into connected components can be used to construct L as a subset of lists of rational numbers, and this definition is purely combinatorial and does not use the subobject classifier Ω. It also follows from the decomposition that L is a distributive lattice with bottom element, i.e. that it has meets: We have

$$\left(\bigcup_{i=1}^{m}(a_i, b_i)\right) \cap \left(\bigcup_{j=1}^{n}(a'_j, b'_j)\right) = \bigcup_{i,j}\left((a_i, b_i) \cap (c_j, d_j)\right)$$

and $(a_i, b_i) \cap (c_j, d_j) = (\max(a_i, c_j), \min(b_i, d_j))$ for all i, j, which is in L.

Definition 2.32. The *cover relation on L* is generated by

$$(a, b) \lhd \left\{ \bigcup_{j=1}^{n} (a'_j, b'_j) \mid a < a'_j, b'_j < b \text{ for } j \leq n \right\}$$

for $a < b$ under binary unions.

Thus $(\bigcup_{i=1}^{m}(a_i, b_i)) \lhd U$ if U is the set of elements $\bigcup_{j=1}^{n}(a'_j, b'_j)$ such that for each j there exists i with $a_i < a'_j$ and $b'_j < b_i$. This cover relation is stable under binary meets and, by definition, joins. It follows that the ω-cpo completion L_ω has enumerable joins and finite meets satisfying the distributivity law. The bottom element of $\emptyset \in L$ is also a bottom element of L_ω. Finally, the subset of elements of L_ω which are bounded by $t = \bigvee_{n \in \mathbb{N}}(-n, n)$ contains the image of L and is closed under joins, hence t is a top element of L_ω. Thus L_ω is a σ-frame.

Definition 2.33. The *σ-frame of formal real opens* $\mathcal{O}(\mathbb{R}_F)$ is given by the ω-cpo completion of L with respect to the covers of Definition 2.32.

By definition, $L \subseteq \mathcal{P}(\mathbb{R})$, but in fact the rational intervals (a, b) are open: Given a Dedekind real $r = (\ell, u) \in \mathbb{R}$, we have $r \in (a, b)$ if and only if $\ell(a) \wedge u(b)$, which is a truth value in $\mathbb{S}$ because $\ell, u : \mathbb{Q} \to \mathbb{S}$. It follows that $L \subseteq \mathcal{O}(\mathbb{R})$. This inclusion is cover preserving because $(a, b) = \bigcup\{(a', b') \mid a < a' \leq b' < b\}$ as subsets of $\mathbb{R}$. We obtain a morphism of ω-cpos $\mathcal{O}(\mathbb{R}_F) \to \mathcal{O}(\mathbb{R})$. It is not necessarily an isomorphism, but it will be assumed for the remainder of this section that it is:

Assumption 2. The map $\mathcal{O}(\mathbb{R}_F) \to \mathcal{O}(\mathbb{R})$ is an isomorphism of partial orders.

Lešnik [59, Section 5.3] proves that if one assumes the intuitionistic principles function-function choice, the continuity principle (which is absurd in classical logic) and the fan principle (every decidable bar is uniform), then every complete metrically separable metric space is metrized. In particular, every open $U \in \mathcal{O}(\mathbb{R})$ is a countable union of metric balls, from which our Assumption 2 follows. Lesnik's assumptions hold in the K2 realizability topos and the big topos of topological spaces, so Assumption 2 holds in these models, too.

Definition and Proposition 2.34. *The map $\lambda' : L \to \mathbb{Q}^+$ given by*

$$\lambda'(\bigcup_{i=1}^{n}(a_i, b_i)) = \sum_{i=1}^{n} b_i - a_i;$$

for $n \geq 0$ and rationals $a_i < b_i \leq a_{i+1}$, $1 \leq i \leq n-1$, is well-defined, monotone and, when coerced to a function $L \to \mathbb{R}_l^+$, cover-preserving. The induced map $\lambda : \mathcal{O}(\mathbb{R}) \cong L_\omega \to \mathbb{R}_l^+$ is a valuation, which we refer to as the Lebesgue valuation.

Proof. λ' is well-defined because decompositions into connected components are unique up to reordering. It is evidently monotone. If $(a, b) \lhd U$, then $(a + n^{-1}, b - n^{-1}) \in U$ for all $n > 0$, so that

$$\bigvee_{u \in U} \lambda'(u) \geq \bigvee_{n>0} \lambda'((a + n^{-1}, b - n^{-1})) = b - a = \lambda'((a, b)).$$

It follows that λ' preserves general covers because we have $\lambda'(x \cup y) = \lambda'(x) + \lambda'(y)$ if x and y are disjoint.

λ preserves the bottom element because λ' does, and it is ω-continuous by construction. What remains to be proved is the modular law

$$\lambda(x \cup y) + \lambda(x \cap y) = \lambda(x) + \lambda(y) \tag{2.7}$$

for all $x, y \in \mathcal{O}(\mathbb{R})$, but we immediately reduce to $x, y \in L$ by induction. In turn, we prove equation (2.7) for $x, y \in L$ by induction over the total number of connected components of x and y. It holds trivially if $x = \emptyset$ or $y = \emptyset$. If $x = (a, b)$ and $y = (c, d)$ are rational intervals, then

$$\begin{aligned} & \lambda(x \cup y) + \lambda(x \cap y) \\ = \; & \max(b, d) - \min(a, c) + \min(b, d) - \max(a, c) \\ = \; & b + d - a - c \\ = \; & \lambda(x) + \lambda(y) \end{aligned}$$

so the equation holds in this case, too.

In the induction step we are given disjoint unions $(a, b) \dot{\cup} x$ and $(c, d) \dot{\cup} y$ such that $b < r$ for all $r \in x$ and $d < s$ for all $d \in y$, at least after reordering the connected components if necessary. If n is the number of connected components of x and m that for y, we may assume that (2.7) holds for all pairs of elements of L whose total number of connected components is at most $n + m + 1$.

Suppose first that (a, b) and (c, d), are disjoint, wlog. say $b \leq c$. Then (a, b) is disjoint from all of x, (c, d) and y, thus

$$\lambda((a, b) \cup x \cup (c, d) \cup y) = \lambda((a, b)) + \lambda(x \cup (c, d) \cup y).$$

By the induction hypothesis,

$$\lambda(x \cup ((c, d) \cup y)) = \lambda(x) + \lambda((c, d) \cup y) - \lambda(x \cap ((c, d) \cup y)).$$

Because (a, b) is disjoint from (c, d) and y, we have

$$((a, b) \cup x) \cap ((c, d) \cup y) = x \cap ((c, d) \cup y),$$

which combined with the previous equations yields the modular law for $(a, b) \cup x$ and $(c, d) \cup y$ if (a, b) and (c, d) are disjoint.

Otherwise (a, b) and (c, d) intersect, so that $(a, b) \cup (c, d) = (e, f)$ with $e = \min(a, c)$ and $f = \max(b, d)$. Without loss of generality we may assume $f = d$, so that all of (a, b), (c, d) and (e, f) are disjoint from y. Thus

$$\begin{aligned} & \lambda((a, b) \cup x \cup (c, d) \cup y) \\ = \; & \lambda(x \cup (e, f) \cup y) \\ = \; & \lambda(x) + \lambda((e, f)) + \lambda(y) - \lambda(x \cap ((e, f) \cup y) \end{aligned}$$

by the induction hypothesis for x and $(e, f) \cup y$. The base case of two rational intervals was already proved, thus

$$\lambda((e, f)) = \lambda((a, b)) + \lambda((c, d)) - \lambda((a, b) \cap (c, d)).$$

Because (a, b) is disjoint from y and λ' maps disjoint unions to sums, we have

$$\lambda(((a, b) \cup x) \cap ((c, d) \cup y)) = \lambda((a, b) \cap (c, d)) + \lambda(x \cap ((c, d) \cup y)).$$

Putting everything together, we obtain

$$\begin{aligned}&\lambda((a,b)\cup x\cup(c,d)\cup y)\\ =\ &\lambda(x)+\lambda((a,b))+\lambda(y)+\lambda((c,d))-\lambda(((a,b)\cup x)\cap((c,d)\cup y))\end{aligned}$$

as required. □

We can now define distributions on (subsets of) the real numbers for which there exists a density with respect to the Lebesgue valuation. For example, the normal distribution $\mathcal{N}(\mu,\sigma^2)$ has density

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\exp\Big(-\frac{1}{2}(\frac{x-\mu}{\sigma})^2\Big),$$

and so $\mathcal{N}(\mu,\sigma^2)\in\mathfrak{G}_{\leq 1}(\mathbb{R})$ can be defined by

$$\mathcal{N}(\mu,\sigma^2)(U)=\int_U f\,d\lambda=\int \mathbb{1}_U f\,d\lambda$$

on opens $U:\mathbb{R}\to\mathbb{S}$.

2.9 Interpreting $\mathcal{R}$ml

The sub-probability Giry monad $\mathfrak{G}_{\leq 1}$ is defined on the cartesian closed category of sets, and the sets of functions $A\to\mathfrak{G}_{\leq 1}(B)$ with the pointwise ordering form ω-cpos with bottom elements. Similarly to Audebaud and Paulin-Mohring [6], we obtain an interpretation of $\mathcal{R}$ml, a call-by-value PCF with recursion and probabilistic choice as effect. Because $\mathfrak{G}_{\leq 1}$ is defined in terms of the intrinsic topology (as opposed to the discrete one), we obtain furthermore an interpretation of primitives for sampling from continuous distributions.

First recall the language PCF as in e.g. Plotkin and Power [68]. We consider the fragment that has as base types N (natural numbers), B (booleans) and R (real numbers), and as type formers finite products and exponentials. Thus the set of types σ is inductively defined by

$$\sigma ::= N\mid B\mid R\mid 1\mid \sigma\times\sigma\mid\sigma\to\sigma.$$

Terms M are given by

$$\begin{aligned}M ::= {}& 0\mid \mathbf{zero}(M)\mid \mathbf{succ}(M)\mid \mathbf{pred}(M)\mid \textbf{nat-to-real}(M)\mid\\ & \mathbf{true}\mid \mathbf{false}\mid \mathbf{if}\ M\ \mathbf{then}\ M\ \mathbf{else}\ M\mid\\ & M<M\mid M+M\mid M\cdot M\mid \exp(M)\mid\log(M)\mid\sin(M)\mid\ldots\mid\\ & \mathbf{bernoulli}\mid\mathbf{uniform}\mid\ldots\mid\\ & *\mid\langle M,M\rangle\mid\pi_1(M)\mid\pi_2(M)\mid\\ & x\mid\lambda x:\sigma.M\mid MM\mid\\ & \mathbf{rec}(f:\sigma\to\tau,x:\sigma.M)\end{aligned}$$

Here f and x can be any one of a fixed set of variables.

There are evident typing rules generating the well-typed terms $\Gamma \vdash M : \sigma$ in a context Γ and rules generating equalities $\Gamma \vdash M_0 = M_1 : \sigma$. (The reduction rules need not concern as here as we are interested in a denotational semantics.) These rules are set up such that $\mathbf{zero}(M) : B$ is well-typed for natural number terms $M : N$ and equal to **true** if and only if $M = 0$. **pred** is the predecessor function (with $\mathbf{pred}(0) = 0$), and **nat-to-real** is the coercion of natural numbers to real numbers. The operators $<, +, -, \cdot$ and $\exp, \log, \sin$ are defined on terms of type R. The list of operators can be expanded as needed with more functions that are constructively definable. Some operators such as the log function are partial; in these cases the semantics of the program is only defined when the operator is applied to a term that is guaranteed to be in the operator's domain.

The terms **bernoulli** : B and **uniform** : R are well-typed in every context Γ; they represent sampling from a fair coin flip and the uniform distribution on the unit interval, respectively. They are typed like constants of type B and R, respectively, and every usage of **bernoulli** and **uniform** samples a fresh value. To reuse sampled values, one must thus bind them to variables.

Much like the list of real operators, the list of distributions can be extended as needed with more constructively definable distributions, for example the normal distribution. Alternatively, many distributions can also be constructed in the language itself if their density with respect to one of the built-in distributions is definable.

The typing rule for the **rec** operator is as follows:

$$\frac{\Gamma, f : \sigma \to \tau, x : \sigma \vdash M : \tau}{\Gamma \vdash \mathbf{rec}(f : \sigma \to \tau, x : \sigma.M) : \sigma \to \tau}$$

rec is used for unbounded recursion and hence satisfies the equation

$$\mathbf{rec}(f : \sigma \to \tau, x : \sigma.M) = \lambda x : \sigma.M[\mathbf{rec}(f : \sigma \to \tau, x : \sigma.M)/f].$$

Plotkin and Power [68] show that cartesian closed categories $\mathcal{C}$ equipped with a suitable monad T can serve as models for call-by-value PCF. In our case, $\mathcal{C} = \mathrm{Set}$ is the category of sets of our ambient constructive logic, and our monad is $T = \mathfrak{G}_{\leq 1}$, the sub-probabilistic Giry monad. Let us verify that Set and $\mathfrak{G}_{\leq 1}$ satisfy the conditions of Plotkin and Power [68]:

- Set is a cartesian closed category enriched over ω-cpos by considering the hom sets as discrete partial orders. It has coproducts (hence an object of booleans) and a natural numbers object.

- The Kleisli category of $\mathfrak{G}_{\leq 1}$ is enriched over ω-cpos with bottom element via the pointwise ordering on maps $A \to \mathfrak{G}_{\leq 1}(B)$. Note that here it is crucial that we work with $\mathfrak{G}_{\leq 1}$ instead of $\mathfrak{G}_{=1}$ as for the latter the hom sets need not have bottom elements.

- The strength of $\mathfrak{G}_{\leq 1}$ preserves bottom elements because the integral $\mathcal{I} \blacktriangleright \mathcal{J}$ of equation (2.6) vanishes if either $\mathcal{I}$ or $\mathcal{J}$ vanishes.

Types are now interpreted as follows:

$$[\![N]\!] = \mathbb{N} \qquad [\![B]\!] = \{0,1\} \qquad [\![R]\!] = \mathbb{R} \qquad [\![1]\!] = 1$$

$$[\![\sigma \times \tau]\!] = [\![\sigma]\!] \times [\![\tau]\!] \qquad [\![\sigma \to \tau]\!] = ([\![\sigma]\!] \to \mathfrak{G}_{\leq 1}([\![\tau]\!]))$$

Contexts $\Gamma = (x_1 : \sigma_1, \dots, x_n : \sigma_n)$ are interpreted as products $[\![\sigma_1]\!] \times \cdots \times [\![\sigma_n]\!]$.

The denotation of a term $\Gamma \vdash M : \sigma$ is a Kleisli arrow $[\![M]\!] : [\![\Gamma]\!] \to \mathfrak{G}_{\leq 1}([\![\sigma]\!])$; the complete set of clauses can be found in Plotkin and Power [68]. For example, the denotation of a successor term $\mathbf{succ}(M)$ for some $\Gamma \vdash M : N$ is defined as composition

$$[\![\Gamma]\!] \xrightarrow{[\![M]\!]} \mathfrak{G}_{\leq 1}(\mathbb{N}) \xrightarrow{\mathfrak{G}_{\leq 1}(\mathrm{succ})} \mathfrak{G}_{\leq 1}(\mathbb{N})$$

using the semantic successor function on the natural numbers object, and the tuple term $\langle M_1, M_2 \rangle$ for terms $\Gamma \vdash M_i : \sigma_i$ is defined as

$$[\![\Gamma]\!] \xrightarrow{\langle [\![M_1]\!], [\![M_2]\!] \rangle} \mathfrak{G}_{\leq 1}([\![\sigma_1]\!]) \times \mathfrak{G}_{\leq 1}([\![\sigma_2)]\!] \xrightarrow{-\blacktriangleright-} \mathfrak{G}_{\leq 1}([\![\sigma_1 \times \sigma_2]\!])$$

using the strength of $\mathfrak{G}_{\leq 1}$.

A recursion term $\Gamma \vdash \mathbf{rec}(f : \sigma \to \tau, x : \sigma.M) : \sigma \to \tau$ for $\Gamma, f : \sigma \to \tau, x : \sigma \vdash M : \tau$ is interpreted as follows. First we define a monotone endofunction Y on the partial order of maps $k : [\![\Gamma, x : \sigma]\!] \to \mathfrak{G}_{\leq 1}([\![\tau]\!])$ as follows. k corresponds to a map $\bar{k} : [\![\Gamma]\!] \to ([\![\sigma]\!] \to \mathfrak{G}_{\leq 1}([\![\tau]\!]))$. Now

$$Y(k) : \ [\![\Gamma, v : \sigma]\!] \xrightarrow{\langle \pi_1, \bar{k} \circ \pi_1, \pi_2 \rangle} [\![\Gamma, f : \sigma \to \tau, v : \sigma]\!] \xrightarrow{[\![M]\!]} \mathfrak{G}_{\leq 1}([\![\tau]\!]).$$

We then define $[\![\mathbf{rec}(f : \sigma \to \tau, x : \sigma.M)]\!]$ as the join of the sequence $\bot \leq Y(\bot) \leq Y(Y(\bot)) \leq \dots$ of functions $[\![\Gamma, v : \sigma]\!] \to \mathfrak{G}_{\leq 1}([\![\tau]\!])$.

The denotations of real functions operators are the corresponding functions on $\mathbb{R}$. The denotation of the comparison operator $<$ is an exception: There is no constructively definable function $\mathbb{R} \times \mathbb{R} \to \{0,1\}$ that decides the ordering of real numbers; in fact such a function would contradict Assumption 2. However, there is a function $l : \mathbb{R} \times \mathbb{R} \to \mathfrak{G}_{\leq 1}(\{0,1\})$ such that

$$l(x_0, x_1) = \begin{cases} \eta(1) & \text{if } x_0 < x_1 \\ \eta(0) & \text{if } x_0 > x_1 \\ \bot & \text{if } x_0 = x_1 \end{cases}$$

and we take l to be the denotation of the $<$ operator. l is can be constructed as follows: As proved by Gilbert [34], there is a map

$$d : \{(a,b) \in \mathbb{S} \times \mathbb{S} \mid a \wedge b = \bot\} \to \{0,1\}_\bot$$

such that $d((\top, \bot)) = 0$ and $d((\bot, \top)) = 1$. Recall that a Dedekind real number x is given by a pair (L, U) of open sets of rational numbers $L, U : \mathbb{Q} \to \mathbb{S}$ satisfying conditions such that $L = \{q \in \mathbb{Q} \mid q < x\}$ and $U = \{q \in \mathbb{Q} \mid q > x\}$. Thus given two real numbers $x_1 = (L_1, U_1)$ an $x_2 = (L_2, U_2)$, it holds that $L_1 \cap U_2$ is inhabited if and only if $x_1 > x_2$, and symmetrically that $L_2 \cap U_1$ is inhabited if and only if $x_1 < x_2$. Inhabitation of open subsets $U \subseteq A$ of countable sets A is an open proposition because it is equivalent to $\bigvee_{a \in A} U(a)$, which is a countable disjunction of open propositions. It follows that inhabitation of $L_1 \cap U_2$ is a proposition $s_1 \in \mathbb{S}$, and similarly inhabitation of $L_2 \cap U_1$ is a proposition $s_2 \in \mathbb{S}$. $s_1 \wedge s_2$ is contradictory. We now set $l(x_1, x_2) = \phi(d(s_1, s_2))$, where ϕ is the canonical map $\{0, 1\}_\bot \to \mathfrak{G}_{\leq 1}(\{0, 1\})$.

The Bernoulli sampling constant $\Gamma \vdash$ **bernoulli** $: B$ is interpreted as the constant function $[\![\Gamma]\!] \to \mathfrak{G}_{\leq 1}([\![B]\!])$ with value the lower integral corresponding to the valuation $\nu(0) = \frac{1}{2}$ and $\nu(1) = \frac{1}{2}$. Similarly, the **uniform** term is interpreted as the lower integral corresponding to $\lambda_{(0,1)}(U) = \lambda(U \cap (0, 1))$ using the Lebesgue valuation of Section 2.8. (Note that we have to use the open interval $(0, 1)$ here instead of the closed interval $[0, 1]$ because $U \cap [0, 1]$ is not always open in $\mathbb{R}$, even if U is open.)

Consider the problem of sampling from a standard normal distribution given only the **uniform** built-in. One way to achieve this using the *Marsaglia polar method* [64] is as follows:

```
let rec normal =
  let
    x = 2 * uniform - 1;
    y = 2 * uniform - 1;
    s = x * x + y * y;
  in
    if s < 1 then x * sqrt ((-2) * ln s / s)
    else normal
```

Note that `normal` is defined by potentially infinite recursion but terminates with probability 1, that is, $[\![\texttt{normal}]\!](\mathbb{R}) = 1 \in \mathbb{R}_l$.

2.10 Conclusion

Contributions. This paper develops the foundations of integration theory in synthetic topology based on the initial σ-frame. The initial σ-frame $\mathbb{S}$ is the ω-cpo completion of the booleans. We discuss several alternative constructions of free ω-cpo completions and show how product ω-cpos can be presented in terms of presentations of their factors. It is shown that $\mathbb{S}$ is a dominance and hence suitable for synthetic topology. Following Escardó and Knapp [30] we show that the $\mathbb{S}$-partial map classifier of a set A is given by its pointed ω-cpo completion $A_\bot$. A set of lower real numbers based on $\mathbb{S}$ is defined and shown to satisfy the universal property of the ω-cpo completion of the rationals. This set of lower reals is then used in definitions of valuations and lower integrals which take into account the intrinsic topology induced by $\mathbb{S}$. The Riesz theorem relating valuations and lower integrals is proved and used to define the Giry monad. The Fubini theorem is shown to hold for sets A, B whose product has the product topology. Finally, the Lebesgue valuation is defined under the assumption of metrizability of $\mathbb{R}$, which would be impossible if our valuations were based on discrete topologies.

Related work. Much of our approach to lower integrals is adapted from Steven Vickers's work [90, 91] work on the same subject, but in the setting of synthetic topology instead of locale theory. Lower integrals are better behaved on locales than in synthetic topology in certain aspects. For example, the Fubini theorem holds without restriction for locales, making the Giry monad commutative, whereas we can only prove the Fubini theorem in synthetic topology on the assumption that the involved products are topologized correctly. On the other hand, the category of locales is not cartesian closed, whereas the ambient category of sets in synthetic topology is even a elementary topos (or, predicatively, a ΠW-pretopos).

Shulman [76, Section 11] proves the Brouwer fixpoint theorem in homotopy type theory using synthetic topology. He uses modalities to mediate between the homotopical and topological circle and other spaces. This spatial (modal) type theory is modelled in any local topos, for example Johnstone's topological topos [49]. Fourman's big topos that models the intuitionistic principles outlined in Section 2.8 is also local. This paper does not focus on homotopy theory, thus the methodology is different.

Escardó and Xu [29] use a similar big topos, but restricted to compact spaces to model the fan-theorem in a simple type theory. Coquand et al. [23] provide a stack model over Cantor space for univalent type theory. It is likely that our work model can be given a constructive treatment by these methods; see Coquand [20].

There is an interesting analogy with the semantics for higher order probabilistic programming described in Heunen et al. [38], Staton et al. [79]. Noting that the category of standard Borel spaces is not Cartesian closed, they embed it into a supercategory (of quasi-Borel spaces) which is closed under exponentials. A similar problem exists in synthetic topology: The category of topological spaces is not Cartesian closed. The common solution is to consider a convenient super-category. Escardó [28, Chapter 10] presents a number of subcategories of presheaves over the category of topological spaces for this purpose. In our case, it is more natural to consider the sheaves for the open cover topology. In this light, one could consider our construction as first embedding in a bigger category with (dependent) function types and then defining the monad on the bigger category. One advantage of semantics in toposes is that they model all of constructive mathematics, including the principle of unique choice. This enables use of a strong internal logic to simplify arguments, as is exemplified in this paper. On the other hand, our Fubini theorem holds only conditionally, whereas it holds for arbitrary products of quasi-Borel spaces, making the Giry monad on quasi-Borel spaces commutative.

Acknowledgements. The questions in this paper originated from discussions with Christine Paulin in 2014, when Spitters held a Digiteo chair at LRI, Inria. We also benefited from Faissole's internship with Paulin about lower reals in Coq. We are grateful for both. We thank Alex Kavvos for discussions on the interpretation of effectful programming and fixed points.

This research was partially supported by the Guarded homotopy type theory project, funded by the Villum Foundation, project number 12386, AFOSR project 'Homotopy Type Theory and Probabilistic Computation', 12595060, and Digiteo.

Chapter 3

The 1-categorical multiverse model

Abstract

Locally cartesian closed (lcc) categories are natural categorical models of extensional dependent type theory. This paper introduces the "gros" semantics in the category of lcc categories: Instead of constructing an interpretation in a given individual lcc category, we show that also the category of all lcc categories can be endowed with the structure of a model of dependent type theory. The original interpretation in an individual lcc category can then be recovered by slicing.

As in the original interpretation, we face the issue of coherence: Categorical structure is usually preserved by functors only up to isomorphism, whereas syntactic substitution commutes strictly with all type theoretic structure. Our solution involves a suitable presentation of the higher category of lcc categories as model category. To that end, we construct a model category of lcc sketches, from which we obtain by the formalism of algebraically (co)fibrant objects model categories of strict lcc categories and then algebraically cofibrant strict lcc categories. The latter is our model of dependent type theory.

3.1 Introduction

Locally cartesian closed (lcc) categories are natural categorical models of extensional dependent type theory [75]: Given an lcc category $\mathcal{C}$, one interprets

- contexts Γ as objects of $\mathcal{C}$;
- (simultaneous) substitutions from context Δ to context Γ as morphisms $f : \Delta \to \Gamma$ in $\mathcal{C}$;
- types $\Gamma \vdash \sigma$ as morphisms $\sigma : \operatorname{dom} \sigma \to \Gamma$ in $\mathcal{C}$ with codomain Γ; and

- terms $\Gamma \vdash s : \sigma$ as sections $s : \Gamma \rightleftarrows \operatorname{dom} \sigma : \sigma$ to the interpretations of types.

A context extension $\Gamma.\sigma$ is interpreted as the domain of σ. Application of a substitution $f : \Delta \to \Gamma$ to a type $\Gamma \vdash \sigma$ is interpreted as pullback

$$\begin{tikzcd} \operatorname{dom} \sigma[f] \arrow[r] \arrow[d, "\sigma[f]"] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{dom} \sigma \arrow[d, "\sigma"] \\ \Delta \arrow[r, "f"] & \Gamma \end{tikzcd}$$

and similarly for terms $\Gamma \vdash s : \sigma$. By definition, the pullback functors $f^* : \mathcal{C}_{/\Gamma} \to \mathcal{C}_{/\Delta}$ in lcc categories $\mathcal{C}$ have both left and right adjoints $\Sigma_f \dashv f^* \dashv \Pi_f$, and these are used for interpreting $\mathbf{\Sigma}$-types and $\mathbf{\Pi}$-types. For example, the interpretation of a pair of types $\Gamma \vdash \sigma$ and $\Gamma.\sigma \vdash \tau$ is a composable pair of morphisms $\Gamma.\sigma.\tau \xrightarrow{\tau} \Gamma.\sigma \xrightarrow{\sigma} \Gamma$, and then the dependent product type $\Gamma \vdash \mathbf{\Pi}_\sigma\, \tau$ is interpreted as $\Pi_\sigma(\tau)$, which is an object of $\mathcal{C}_{/\Gamma}$, i.e. a morphism into Γ.

However, there is a slight mismatch: Syntactic substitution is functorial and commutes strictly with type formers, whereas pullback is generally only pseudo-functorial and hence preserves universal objects only up to isomorphism. Here functoriality of substitution means that if one has a sequence $\mathcal{E} \xrightarrow{g} \Gamma \xrightarrow{f} \Delta$ of substitutions, then we have equalities $\sigma[g][f] = \sigma[gf]$ and $s[g][f] = s[gf]$, i.e. substituting in succession yields the same result as substituting with the composition. For pullback functors, however, we are only guaranteed a natural isomorphism $f^* \circ g^* \cong (g \circ f)^*$. Similarly, in type theory we have $(\mathbf{\Pi}_\sigma\, \tau)[f] = \mathbf{\Pi}_{\sigma[f]}\, \tau[f^+]$ (where f^+ denotes the weakening of f along σ), whereas for pullback functors there merely exist isomorphisms $f^*(\Pi_\sigma(\tau)) \cong \Pi_{f^*(\sigma)}\, (f^+)^*(\tau)$.

In response to these problems, several notions of models with strict pullback operations were introduced, e.g. categories with families (cwfs) [27], and coherence techniques were developed to “strictify” weak models such as lcc categories to obtain models with well-behaved substitution [24, 40, 61]. Thus to interpret dependent type theory in some lcc category $\mathcal{C}$, one first constructs an equivalence $\mathcal{C} \simeq \mathcal{C}^s$ such that $\mathcal{C}^s$ can be endowed with the structure of a strict model of type theory (say, cwf structure), and then interprets type theory in $\mathcal{C}^s$.

In this paper we construct cwf structure on the category of all lcc categories instead of cwf structure on some specific lcc category. First note that the classical interpretation of type theory in an lcc category $\mathcal{C}$ is essentially an interpretation in the slice categories of $\mathcal{C}$:

- Objects $\Gamma \in \operatorname{Ob} \mathcal{C}$ can be identified with slice categories $\mathcal{C}_{/\Gamma}$.
- Morphisms $f : \Delta \to \Gamma$ can be identified with lcc functors $f^* : \mathcal{C}_{/\Gamma} \to \mathcal{C}_{/\Delta}$ which commute with the pullback functors $\Gamma^* : \mathcal{C} \to \mathcal{C}_{/\Gamma}$ and $\Delta^* : \mathcal{C} \to \mathcal{C}_{/\Delta}$.

- Morphisms $\sigma : \operatorname{dom}\sigma \to \Gamma$ with codomain Γ can be identified with the objects of the slice categories $\mathcal{C}_{/\Gamma}$.
- Sections $s : \Gamma \leftrightarrows \operatorname{dom}\sigma : \sigma$ can be identified with morphisms $1 \to \sigma$ with $1 = \mathrm{id}_\Gamma$ the terminal object in the slice category $\mathcal{C}_{/\Gamma}$.

Removing all reference to the base category $\mathcal{C}$, we may now attempt to interpret

- each context Γ as a separate lcc category;
- a substitution from Δ to Γ as an lcc functor $f : \Gamma \to \Delta$;
- types $\Gamma \vdash \sigma$ as objects $\sigma \in \operatorname{Ob}\Gamma$; and
- terms $\Gamma \vdash s : \sigma$ as morphisms $s : 1 \to \sigma$ from a terminal object 1 to σ.

In the original interpretation, substitution in types and terms is defined by the pullback functor $f^* : \mathcal{C}_{/\Gamma} \to \mathcal{C}_{/\Delta}$ along a morphism $f : \Delta \to \Gamma$. In our new interpretation, f is already an lcc functor, which we simply apply to objects and morphisms of lcc categories.

The idea that different contexts should be understood as different categories is by no means novel, and indeed widespread among researchers of geometric logic; see e.g. Vickers [88, section 4.5]. Not surprisingly, some of the ideas in this paper have independently already been explored, in more explicit form, in Vickers [92] for geometric logic. To my knowledge, however, an interpretation of type theory along those lines, especially one with strict substitution, has never been spelled out explicitly, and the present paper is an attempt at filling this gap.

Like Seely's original interpretation, the naive interpretation in the category of lcc categories outlined above suffers from coherence issues: Lcc functors preserve lcc structure up to isomorphism, but not necessarily up to equality, and the latter would be required for a model of type theory.

Even worse, our interpretation of contexts as lcc categories does not admit well-behaved context extensions. Recall that for a context Γ and a type $\Gamma \vdash \sigma$ in a cwf, a context extension consists of a context morphism $p : \Gamma.\sigma \to \Gamma$ and a term $\Gamma.\sigma \vdash v : \sigma[p]$ such that for every morphism $f : \Delta \to \Gamma$ and term $\Delta \vdash s : \sigma[f]$ there is a unique morphism $\langle f, s\rangle : \Delta \to \Gamma.\sigma$ over Γ such that $v[\langle f, s\rangle] = s$. In our case a context morphism is an lcc functor in the opposite direction. Thus a context extension of an lcc category Γ by $\sigma \in \operatorname{Ob}\Gamma$ would consist of an lcc functor $p : \Gamma \to \Gamma.\sigma$ and a morphism $v : 1 \to p(\sigma)$ in $\Gamma.\sigma$, and (p, v) would have to be suitably initial. At first sight it might seem that the slice category $\Gamma_{/\sigma}$ is a good candidate: Pullback along the unique map $\sigma \to 1$ defines an lcc functor $\sigma^* : \Gamma \cong \Gamma_{/1} \to \Gamma_{/\sigma}$. The terminal object of $\Gamma_{/\sigma}$ is the identity on σ, and applying σ^* to σ itself yields the first projection $\mathrm{pr}_1 : \sigma \times \sigma \to \sigma$. Thus the diagonal $d : \sigma \to \sigma \times \sigma$ is a term $\Gamma_{/\sigma} \vdash d : \sigma^*(\sigma)$. The problem is that, while this data is indeed universal, it is only so in the bicategorical sense (see Lemma 3.29): Given an lcc functor $f : \Gamma \to \Delta$ and term $\Delta \vdash w : f(\sigma)$, we obtain an lcc functor

$$\langle f, s\rangle : \Gamma_{/\sigma} \xrightarrow{f_{/\sigma}} \Delta_{/f(\sigma)} \xrightarrow{s^*} \Delta,$$

however, $\langle f, s\rangle$ commutes with σ^* and f only up to natural isomorphism, the equation $\langle f, s\rangle(d) = s$ holds only up to this natural isomorphism, and $\langle f, s\rangle$ is unique only up to unique isomorphism.

The issue with context extensions can be understood from the perspective of comprehension categories, an alternative notion of model of type theory, as follows. Our cwf is constructed on the opposite of Lcc, the category of lcc categories and lcc functors.[1] The corresponding comprehension category should thus consist of a Grothendieck fibration $p : \mathcal{E} \to \mathrm{Lcc}^{\mathrm{op}}$ and a functor $\mathcal{P} : \mathcal{E} \to (\mathrm{Lcc}^{\mathrm{op}})^{\to}$ to the arrow category of $\mathrm{Lcc}^{\mathrm{op}}$ such that

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "\mathcal{P}"] \arrow[dr, "p"'] & & (\mathrm{Lcc}^{\mathrm{op}})^{\to} \arrow[dl, "\mathrm{cod}"] \\
& \mathrm{Lcc}^{\mathrm{op}} &
\end{tikzcd}$$

commutes and $\mathcal{P}(f)$ is a pullback square for each cartesian morphism f in $\mathcal{E}$. The data of a Grothendieck fibration p as above is equivalent to a (covariant) functor $\mathrm{Lcc} \to \mathrm{Cat}$ via the Grothendieck construction, and here we simply take the forgetful functor. Thus the objects of $\mathcal{E}$ are pairs (Γ, τ) such that Γ is an lcc category and $\tau \in \mathrm{Ob}\,\Gamma$, and a morphism $(\Gamma, \tau) \to (\Delta, \sigma)$ in $\mathcal{E}$ is a pair (f, k) of lcc functor $f : \Delta \to \Gamma$ and morphism $k : \tau \to f(\sigma)$ in Γ.

The functor $\mathcal{P}$ should assign to objects (Γ, τ) of $\mathcal{E}$ the projection of the corresponding context extension, hence we define $\mathcal{P}(\Gamma, \tau) = \tau^* : \Gamma \to \Gamma_{/\tau}$ as the pullback functor to the slice category. The cartesian morphisms of $\mathcal{E}$ are those with invertible vertical components k, so they are given up to isomorphism by pairs of the form $(f, \mathrm{id}) : (\Gamma, f(\sigma)) \to (\Delta, \sigma)$. The images of such morphisms under $\mathcal{P}$ are squares

$$\begin{tikzcd}
\Delta \arrow[r, "f"] \arrow[d, "\sigma^*"] & \Gamma \arrow[d, "f(\tau)^*"] \\
\Delta_{/\sigma} \arrow[r, "f_{/\sigma}"] & \Gamma_{/f(\tau)}
\end{tikzcd} \tag{3.1}$$

in Lcc. For $(p, \mathcal{P})$ to be a comprehension category, they would have to be pushout squares, but in fact they are bipushout squares: They satisfy the universal property of pushouts up to unique isomorphism, but not up to equality. If f does not preserve pullback squares up to strict equality, then the

[1]Not to be confused with the category Lcc of lcc sketches of Definition 3.6; here we mean the category of fully realized lcc categories, i.e. fibrant lcc sketches.

square (3.1) commutes only up to isomorphism, not equality. Thus $\mathcal{P}$ is not even a functor but a bifunctor.

Usually when one considers coherence problems for type theory, the problem lies in the fibration p, which is often not strict, and it suffices to change the total category $\mathcal{E}$ while leaving the base unaltered. Our fibration p is already strict, but it does not support substitution stable type constructors. Here the main problem is the base category, however: The required pullback diagrams exist only in the bicategorical sense. Thus the usual constructions [40, 61] are not applicable.

The goal must thus be to find a category that is (bi)equivalent to Lcc in which we can replace the bipushout squares (3.1) by 1-categorical pushouts. Our tool of choice to that end will be *model category theory* (see e.g. Hirschhorn [39]). Model categories are presentations of higher categories as ordinary 1-categories with additional structure. Crucially, model categories allow the computation of higher (co)limits as ordinary 1-categorical (co)limits under suitable hypotheses. The underlying 1-category of the model category presenting a higher category is not unique, and some presentations are more suitable for our purposes than others. We explore three Quillen equivalent model categories, all of which encode the same higher category of lcc categories, and show that the third one indeed admits the structure of a model of dependent type theory.

Because of its central role in the paper, the reader is thus expected to be familiar with some notions of model category theory. We make extensive use of the notion of algebraically (co)fibrant object in a model category [17, 66], but the relevant results are explained where necessary and can be taken as black boxes for the purpose of this paper. Because of the condition on enrichment in Theorem 3.21, all model categories considered here are proved to be model Gpd-categories, that is, model categories enriched over the category of groupoids with their canonical model structure. See Guillou and May [36] for background on enriched model category theory, Anderson [4] for the canonical model category of groupoids, and Lack [58] for the closely related model Cat-categories. While it is more common to work with the more general simplicially enriched model categories, the fact that the higher category of lcc categories is 2-truncated affords us to work with simpler groupoid enrichments instead.

In Section 3.2 we construct the model category Lcc of *lcc sketches*, a left Bousfield localization of an instance of Isaev's model category structure on marked objects [47]. Lcc sketches are to lcc categories as finite limit sketches are to finite limit categories. Thus lcc sketches are categories with some diagrams marked as supposed to correspond to a universal object of lcc categories, but marked diagrams do not have to actually satisfy the universal property. The model category structure is set up such that every lcc sketch generates an lcc category via fibrant replacement, and lcc sketches are equivalent if and only if they generate equivalent lcc categories.

In Section 3.3 we define the model category sLcc of *strict lcc categories*. Strict lcc categories are the algebraically fibrant objects of Lcc, that is, they are objects of Lcc *equipped* with canonical lifts against trivial cofibrations witnessing their fibrancy in Lcc. Such canonical lifts correspond to canonical choices of universal objects in lcc categories, and the morphisms in sLcc preserve these canonical choices not only up to isomorphism but up to equality.

Section 3.4 finally establishes the model of type theory in the opposite of Coa sLcc, the model category of *algebraically cofibrant objects* in sLcc. The objects of Coa sLcc are strict lcc categories Γ such that every (possibly non-strict) lcc functor $\Gamma \to \Delta$ has a canonical strict isomorph. This additional structure is crucial to reconcile the context extension operation, which is given by freely adjoining a morphism to a strict lcc category, with taking slice categories.

In Section 3.5 we show that the cwf structure on $(\mathrm{Coa}\,\mathrm{sLcc})^{\mathrm{op}}$ can be used to rectify Seely's original interpretation in a given lcc category $\mathcal{C}$. This is done by choosing an equivalent lcc category $\Gamma \simeq \mathcal{C}$ with $\Gamma \in \mathrm{Ob}\,(\mathrm{Coa}\,\mathrm{sLcc})$, and then Γ inherits cwf structure from the core of the slice cwf $(\mathrm{Coa}\,\mathrm{sLcc})^{\mathrm{op}}_{/\Gamma}$.

Acknowledgements. This paper benefited significantly from input by several members of the research community. I would like to thank the organizers of the TYPES workshop 2019 and the HoTT conference 2019 for giving me the opportunity to present preliminary versions of the material in this paper. Conversations with Emily Riehl, Karol Szumiło and David White made me aware that the constructions in this paper can be phrased in terms of model category theory. Daniel Gratzer pointed out to me the biuniversal property of slice categories. Valery Isaev explained to me some aspects of the model category structure on marked objects. I would like to thank my advisor Bas Spitters for his advice on this paper, which is part of my PhD research.

This work was supported by the Air Force Office and Scientific Research project "Homotopy Type Theory and Probabilistic Computation", grant number 12595060.

3.2 Lcc sketches

This section is concerned with the model category Lcc of lcc sketches. Lcc is constructed as the left Bousfield localization of a model category of lcc-marked objects, an instance of Isaev's model category structure on marked objects.

Definition 3.1 (Isaev [47] Definition 2.1)**.** Let $\mathcal{C}$ be a category and let $i : I \to \mathcal{C}$ be a diagram in $\mathcal{C}$. An *(i-)marked object* is given by an object X in $\mathcal{C}$ and a subfunctor m_X of $\mathrm{Hom}(i(-), X) : I^{\mathrm{op}} \to \mathrm{Set}$. A map of the form $k : i(K) \to X$ is *marked* if $k \in m_X(K)$.

A morphism of i-marked objects is a marking-preserving morphism of underlying objects in $\mathcal{C}$, i.e. a morphism $f : X \to Y$ such that the image of m_X under postcomposition by f is contained in m_Y. The category of i-marked objects is denoted by $\mathcal{C}^i$.

The forgetful functor $U : \mathcal{C}^i \to \mathcal{C}$ has a left and right adjoint: Its left adjoint $X \mapsto X^\flat$ is given by equipping an object X of $\mathcal{C}$ with the minimal marking $m_{X^\flat} = \emptyset \subseteq \mathrm{Hom}(i(-), X)$, while the right adjoint $X \mapsto X^\sharp$ equips objects with their maximal marking $m_{X^\sharp} = \mathrm{Hom}(i(-), X)$.

In our application, $\mathcal{C} = \mathrm{Cat}$ is the category of (sufficiently small) categories, and $I = I_{\mathrm{lcc}}$ contains diagrams corresponding to the shapes (e.g. a squares for pullbacks) of lcc structure.

Definition 3.2. The subcategory $I_{\mathrm{lcc}} \subseteq \mathrm{Cat}$ of *lcc shapes* is given as follows: Its objects are the three diagrams Tm, Pb and Pi. Tm is given by the category with a single object t and no nontrivial morphisms; it corresponds to terminal objects. Pb is the free-standing non-commutative square

$$\begin{tikzcd} \cdot \arrow[r, "p_2"] \arrow[d, "p_1"] & \cdot \arrow[d, "f_2"] \\ \cdot \arrow[r, "f_1"] & \cdot; \end{tikzcd}$$

and corresponds to pullback squares. Pi is the free-standing non-commutative diagram

$$\begin{tikzcd} \cdot \arrow[dr, "\varepsilon"] \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\ & \cdot \arrow[d, "g"] & \cdot \arrow[d, "f_2"] \\ & \cdot \arrow[r, "f_1"] & \cdot \end{tikzcd}$$

and corresponds to dependent products $f_2 = \Pi_{f_1}(g)$ and their evaluation maps ε. The only nontrivial functor in I_{lcc} is the inclusion of Pb into Pi as indicated by the variable names. It corresponds to the requirement that the domain of the evaluation map of dependent products must be a suitable pullback.

We obtain the category $\mathrm{Cat}^{\mathrm{lcc}} = \mathrm{Cat}^{I_{\mathrm{lcc}}}$ of *lcc-marked categories.*

Now suppose that $\mathcal{C} = \mathcal{M}$ is a model category. Let $\gamma : \mathcal{M} \to \mathrm{Ho}\,\mathcal{M}$ be the quotient functor to the homotopy category. A marking $m_X \subseteq \mathrm{Hom}(i(-), X)$ of some $X \in \mathrm{Ob}\,\mathcal{M}$ induces a canonical marking $\gamma(m_X) \subseteq \mathrm{Hom}(\gamma(i(-)), \gamma(X))$ on $\gamma(X)$ by taking $\gamma(m_X)$ to be the image of m_X under γ. Thus a morphism $K \to X$ in $\mathrm{Ho}\,\mathcal{M}$ is marked if and only if it has a preimage under γ which is marked.

Theorem 3.3 (Isaev [47] Theorem 3.3)**.** *Let $\mathcal{M}$ be a combinatorial model category and let $i : I \to \mathcal{M}$ be a diagram in $\mathcal{M}$ such that every object in the image of i is cofibrant. Then the following defines the structure of a combinatorial model category on $\mathcal{M}^i$:*

- *A morphism $f : (m_X, X) \to (m_Y, Y)$ in $\mathcal{M}^i$ is a cofibration if and only if $f : X \to Y$ is a cofibration in $\mathcal{M}$.*

- *A morphism $f : (m_X, X) \to (m_Y, Y)$ in $\mathcal{M}^i$ is a weak equivalence if and only if $\gamma(f) : (\gamma(m_X), \gamma(X)) \to (\gamma(m_Y), \gamma(Y))$ is an isomorphism in $(\operatorname{Ho}\mathcal{M})^{\gamma i}$.*

A marked object (X, m_X) is fibrant if and only if X is fibrant in $\mathcal{M}$ and the markings of X are stable under homotopy; that is, if $k \simeq h : i(K) \to X$ are homotopic maps in $\mathcal{M}$ and k is marked, then h is marked. The adjunctions $(-)^\flat \dashv U$ and $U \dashv (-)^\sharp$ are Quillen adjunctions.

Remark 3.4. The description of weak equivalences in Theorem 3.3 does not appear as stated in Isaev [47], but follows easily from results therein. Let $t : \mathrm{Id} \Rightarrow R : \mathcal{M} \to \mathcal{M}$ be a fibrant replacement functor. By Isaev [47, lemma 2.5], a map $f : (m_X, X) \to (m_Y, Y)$ is a weak equivalence in $\mathcal{M}^i$ if and only if f is a weak equivalence in $\mathcal{M}$ and for every diagram (of solid arrows)

$$\begin{tikzcd}
i(K) \arrow[dr, dashed, "h'"] \arrow[drr, bend left, "k"] \arrow[ddr, bend right, "h"'] \arrow[ddr, phantom, "\simeq"] & & \\
& X \arrow[r, "f"] \arrow[d, "t_X"] & Y \arrow[d, "t_Y"] \\
& R(X) \arrow[r, "R(f)"] & R(Y)
\end{tikzcd} \tag{3.2}$$

in which the outer square commutes up to homotopy and k is marked, there exists a marked map $h' : i(K) \to X$ as indicated such that $t_X h' \simeq h$. (h' is not required to commute with k and f.)

Now assume that $f : (m_X, X) \to (m_Y, Y)$ satisfies this condition and let us prove that $\gamma(f)$ is an isomorphism of induced marked objects in the homotopy category. $\gamma(f)$ is an isomorphism in $\operatorname{Ho}\mathcal{M}$, so it suffices to show that $\gamma(f)^{-1}$ preserves markings. By definition, every marked morphism of $\gamma(Y)$ is of the form $\gamma(k) : \gamma(i(K)) \to \gamma(Y)$ for some marked $k : i(K) \to Y$. Because $i(K)$ is cofibrant and $R(X)$ is fibrant, the map $\gamma(t_X) \circ \gamma(f)^{-1} \circ \gamma(k) : \gamma(i(K)) \to \gamma(R(X))$ has a preimage $h : i(K) \to R(X)$ under γ. As $i(K)$ is cofibrant, $R(Y)$ is fibrant and $\gamma(R(f) \circ h) = \gamma(t_Y k)$, there is a homotopy $h \circ R(f) \simeq t_Y k$. By assumption, there exists a marked map $h' : i(K) \to X$ such that $h' t_X \simeq h$, thus $\gamma(f)^{-1} \circ \gamma(k) = \gamma(h')$ is marked.

To prove the other direction of the equivalence, assume that $\gamma(f)$ is an isomorphism of marked objects and let h, k be as in diagram (3.2). $\gamma(f)^{-1}\gamma(k)$ is marked, hence has a preimage $h' : i(K) \to X$ under γ which is marked. We have $\gamma(t_X h') = \gamma(h)$ because postcomposition of both sides with the isomorphism $\gamma(R(f))$ gives equal results. $i(K)$ is cofibrant and $R(X)$ is fibrant, thus $t_X h' \simeq h$.

Lemma 3.5. *Let $\mathcal{M}$ and $i : K \to \mathcal{M}$ be as in Theorem 3.3.*

(1) If $\mathcal{M}$ is a left proper model category, then $\mathcal{M}^i$ is a left proper model category.

(2) *If $\mathcal{M}$ is a model* Gpd*-category, then $\mathcal{M}^i$ admits the structure of a model* Gpd*-category such that $(-)^\flat \dashv U$ and $U \dashv (-)^\sharp$ lift to Quillen* Gpd*-adjunctions.*

Proof. (1). Let

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & Y_2 \arrow[d, "f'"] \\ Y_1 \arrow[r, "g'"'] \arrow[ur, phantom, "\ulcorner"] & Z \end{tikzcd}$$

be a pushout square in $\mathcal{M}^i$ such that f is a weak equivalence. $\mathcal{M}$ is left proper, so $\gamma(f')$ is invertible as a map in $\operatorname{Ho}\mathcal{M}$.

A map $k : i(K) \to U(Z)$ is marked if and only if it factors via a marked map $k_1 : i(K) \to U(Y_1)$ or via a marked map $k_2 : i(K) \to U(Y_2)$. In the first case,

$$\gamma(f')^{-1} \circ \gamma(k) = \gamma(g) \circ \gamma(f)^{-1} \circ \gamma(k_1),$$

which is marked because f is a weak equivalence. Otherwise

$$\gamma(f')^{-1}\gamma(k) = \gamma(k_2),$$

which is also marked. We have shown that $\gamma(f')$ is an isomorphism of marked objects in $\operatorname{Ho}\mathcal{M}$, thus f' is a weak equivalence.

(2). Let X and Y be marked objects. We define the mapping groupoid $\mathcal{M}^i(X, Y)$ as the full subgroupoid of $\mathcal{M}(U(X), U(Y))$ of marking preserving maps.

$\mathcal{M}^i$ is complete and cocomplete as a 1-category. Thus if we construct tensors $\mathcal{G} \otimes X$ and powers $X^{\mathcal{G}}$ for all $X \in \operatorname{Ob}\mathcal{M}$ and $\mathcal{G} \in \operatorname{Ob}\mathrm{Gpd}$ it follows that $\mathcal{M}^i$ is also complete and cocomplete as a Gpd-category. The underlying object of powers and copowers is constructed in $\mathcal{M}$, i.e. $G(\mathcal{G} \otimes X) = \mathcal{G} \otimes G(X)$ and $G(X^{\mathcal{G}}) = G(X)^{\mathcal{G}}$. A map $k : i(K) \to X^{\mathcal{G}}$ is marked if and only if the composite

$$i(K) \xrightarrow{k} G(X)^{\mathcal{G}} \xrightarrow{X^v} G(X)^1 = G(X)$$

is marked for every $v \in \operatorname{Ob}\mathcal{G}$ (which we identify with a map $v : 1 \to \mathcal{G}$). Similarly, a map $k : i(K) \to \mathcal{G} \otimes X$ is marked if and only if it factors as

$$i(K) \xrightarrow{k_0} G(X) = 1 \otimes G(X) \xrightarrow{v \otimes \mathrm{id}} \mathcal{G} \otimes G(X)$$

for some object v in $\mathcal{G}$ and marked k_0. It follows by Kelly and Kelly [57, Theorem 4.85] from the preservation of tensors and powers by U that the 1-categorical adjunctions $(-)^\flat \dashv U$ and $U \dashv (-)^\flat$ extend to Gpd-adjunctions.

It remains to show that the tensoring $\mathrm{Gpd} \times \mathcal{M}^i \to \mathcal{M}^i$ is a Quillen bifunctor. For this we need to prove that if $f : \mathcal{G} \rightarrowtail \mathcal{H}$ is a cofibration of groupoids and $g : X \rightarrowtail Y$ is a cofibration of marked objects, then their pushout-product

$$f \,\square\, g : \mathcal{G} \otimes Y \amalg_{\mathcal{G} \otimes X} \mathcal{H} \otimes X \to \mathcal{H} \otimes Y$$

is a cofibration, and that it is a weak equivalence if either f or g is furthermore a weak equivalence. The first part follows directly from the same property for the Gpd-enrichment of $\mathcal{M}$ and the fact that U preserves tensors and pushouts, and reflects cofibrations.

In the second part we have in both cases that $f \,\square\, g$ is a weak equivalence in $\mathcal{M}$. Thus we only need to show that $f \,\square\, g$ reflects a given marked morphism $k : i(K) \to \mathcal{H} \otimes G(Y)$ in $\mathrm{Ho}\,\mathcal{M}$. It follows from the construction of $\mathcal{H} \otimes Y$ that for any such k there exists $w \in \mathrm{Ob}\,\mathcal{H}$ such that $k = (w \otimes \mathrm{id}) \circ k_0$ for some marked map $k_0 : i(K) \to G(Y)$.

Assume first that f is a trivial cofibration, i.e. an equivalence of groupoids that is injective on objects. Then there exists $v \in \mathrm{Ob}\,\mathcal{G}$ such that $f(v)$ and w are isomorphic objects of $\mathcal{H}$. k is (left) homotopic to $(f(v) \otimes \mathrm{id}) \circ k_0$, which factors via the marked map $(v \otimes \mathrm{id}) \circ k_0 : i(K) \to \mathcal{G} \otimes G(Y)$. It follows that $\gamma(f \,\square\, g)$ reflects marked morphisms.

Now assume that g is a trivial cofibration. Then $\gamma(g)$ reflects marked maps, i.e. there exists a marked map $h_0 : i(K) \to G(X)$ such that $\gamma(g) \circ \gamma(h_0) = \gamma(h)$. Thus the equivalence class of $(w \otimes \mathrm{id}) \circ h_0 : i(K) \to \mathcal{H} \otimes X$ in $\mathrm{Ho}\,\mathcal{M}$ is marked and mapped to $\gamma(k)$ under postcomposition by $\gamma(f)$. $\square$

In the semantics of logic, one usually defines the notion of model of a logical theory in two steps: First a notion of structure is defined that interprets the theory's signature, i.e. the function and relation symbols that occur in its axioms. Then one defines what it means for such a structure to satisfy a formula over the signature, and a model is a structure of the theory's signature which satisfies the theory's axioms. For very well-behaved logics such as Lawvere theories, there is a method of freely turning structures into models of the theory, so that the category of models is a reflective subcategory of the category of structures.

By analogy, lcc-marked categories correspond to the structures of the signature of lcc categories. The model structure of $\mathrm{Cat}^{\mathrm{lcc}}$ ensures that markings respect the homotopy theory of Cat, in that the choice of marking is only relevant up to isomorphism of diagrams. However, the model structure does not encode the universal property that marked diagrams are ultimately supposed to satisfy. To obtain the analogue of the category of models for a logical theory, we now define a reflective subcategory of $\mathrm{Cat}^{\mathrm{lcc}}$. The technical tool to accomplish this is a left Bousfield localization at a set S of morphisms in $\mathrm{Cat}^{\mathrm{lcc}}$. S corresponds to the set of axioms of a logical theory. We thus need to define S in such a way that an lcc-marked category is lcc if and only if it has the right lifting property against the morphisms in S such that lifts are determined uniquely up to unique isomorphism.

Cat is a combinatorial and left proper model Gpd-category with mapping groupoids $\mathrm{Cat}(\mathcal{C}, \mathcal{D})$ given by sets of functors and their natural isomorphisms. Thus $\mathrm{Cat}^{\mathrm{lcc}}$ has the structure of a combinatorial and left proper model Gpd-category by Lemma 3.5. It follows that the left Bousfield localization at any (small) set of maps exists by Hirschhorn [39, Theorem 4.1.1].

Definition 3.6. The model category Lcc of *lcc sketches* is the left Bousfield localization of the model category of lcc-marked categories at the following morphisms.

- The morphism tm_1 given by the unique map from the empty category to the marked category with a single, Tm-marked object. tm_1 corresponds to the essentially unique existence of a terminal object.
- The morphism tm_2 given by the inclusion of the category with two objects

  $$\cdot \qquad t$$

  such that t is Tm-marked into

  $$\cdot \longrightarrow t.$$

  tm_2 corresponds to the universal property of terminal objects.
- The morphism pb_0 given by the quotient map from the free-standing non-commutative and Pb-marked square

  $$\begin{tikzcd} \cdot \arrow[r, "p_2"] \arrow[d, "p_1"] & \cdot \arrow[d, "f_2"] \\ \cdot \arrow[r, "f_1"] & \cdot \end{tikzcd}$$

  to the commuting square

  $$\begin{tikzcd} \cdot \arrow[r, "p_2"] \arrow[d, "p_1"] \arrow[dr, phantom, "\circlearrowleft"] & \cdot \arrow[d, "f_2"] \\ \cdot \arrow[r, "f_1"] & \cdot \end{tikzcd}$$

  (which is still marked via Pb). pb_0 corresponds to the commutativity of pullback squares.
- The morphism pb_1 given by the inclusion of the cospan

  $$\begin{tikzcd} & \cdot \arrow[d, "f_2"] \\ \cdot \arrow[r, "f_1"] & \cdot \end{tikzcd}$$

  with no markings into the non-commutative square

  $$\begin{tikzcd} \cdot \arrow[r, "p_2"] \arrow[d, "p_1"] & \cdot \arrow[d, "f_2"] \\ \cdot \arrow[r, "f_1"] & \cdot \end{tikzcd}$$

  which is marked via Pb. pb_1 corresponds to the essentially unique existence of pullback squares.

- The morphism pb_2 given by the inclusion of

$$\begin{tikzcd}
\cdot \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] \arrow[dr, phantom, "\circlearrowleft"] & & \\
& \cdot \arrow[r, "p_2"] \arrow[d, "p_1"] & \cdot \arrow[d, "f_2"] \\
& \cdot \arrow[r, "f_1"] & \cdot
\end{tikzcd}$$

  in which the lower right square is non-commutative and marked via Pb, into the diagram

$$\begin{tikzcd}
\cdot \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] \arrow[dr] \arrow[dr, phantom, shift left=3, "\circlearrowleft"] \arrow[dr, phantom, shift right=3, "\circlearrowleft"'] & & \\
& \cdot \arrow[r, "p_2"] \arrow[d, "p_1"] & \cdot \arrow[d, "f_2"] \\
& \cdot \arrow[r, "f_1"] & \cdot
\end{tikzcd}$$

  in which the indicated triangles commute. pb_2 corresponds to the universal property of pullback squares.

- The morphism pi_0 given by the quotient map of the non-commutative diagram

$$\begin{tikzcd}
\cdot \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] \arrow[dr, "\varepsilon"] & & \\
& \cdot \arrow[d, "g"] & \cdot \arrow[d, "f_2"] \\
& \cdot \arrow[r, "f_1"] & \cdot
\end{tikzcd}$$

  in which the square made of the p_i and f_i is marked via Pb and the whole diagrams is marked via Pi, to

$$\begin{tikzcd}
\cdot \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] \arrow[dr, "\varepsilon"] \arrow[dr, phantom, shift right=3, "\circlearrowleft"'] & & \\
& \cdot \arrow[d, "g"] & \cdot \arrow[d, "f_2"] \\
& \cdot \arrow[r, "f_1"] & \cdot
\end{tikzcd}$$

  in which the indicated triangle commutes. pi_0 corresponds to the requirement that the evaluation map ε of the dependent product $f_2 = \Pi_{f_1}(g)$ is a morphism in the slice category over $\operatorname{cod} g$.

- The morphism pi_1 given by the inclusion of a composable pair of morphisms

$$\begin{tikzcd}
\cdot \arrow[d, "g"] & \\
\cdot \arrow[r, "f_1"] & \cdot
\end{tikzcd}$$

into the non-commutative diagram

$$
\begin{tikzcd}
\cdot \arrow[dr, "\varepsilon"] \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\
& \cdot \arrow[d, "g"] & \cdot \arrow[d, "f_2"] \\
& \cdot \arrow[r, "f_1"] & \cdot
\end{tikzcd}
$$

which is marked via Pi (and hence the outer square is marked via Pb). pi_1 corresponds to the essentially unique existence of dependent products $f_2 = \Pi_{f_1}(g)$ and their evaluation maps ε.

- The morphism pi_2 given by the inclusion of the diagram

$$
\begin{tikzcd}
\cdot \arrow[ddrrrr, bend left, "p_2'"] \arrow[ddrr, bend right, "e"] \arrow[dddrr, bend right, "p_1'"'] & & & & \\
& \cdot \arrow[dr, "\varepsilon"] \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & & \\
& & \cdot \arrow[d, "g"] & \cdot \arrow[d, "f_2"] & \cdot \arrow[dl, "f_2'"] \\
& & \cdot \arrow[r, "f_1"] & \cdot &
\end{tikzcd}
$$

in which the square given by the f_i and p_i is marked via Pb, the subdiagram given by the f_i, p_i, g and ε is marked via Pi, the square given by f_1, f_2' and the p_i' is marked via Pb, and $e \circ g = p_1'$, into the diagram

$$
\begin{tikzcd}
\cdot \arrow[dr, "u"] \arrow[ddrrrr, bend left, "p_2'"] \arrow[ddrr, bend right, "e"] \arrow[dddrr, bend right, "p_1'"'] & & & & \\
& \cdot \arrow[dr, "\varepsilon"] \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & & \\
& & \cdot \arrow[d, "g"] & \cdot \arrow[d, "f_2"] & \cdot \arrow[l] \arrow[dl, "f_2'"] \\
& & \cdot \arrow[r, "f_1"] & \cdot &
\end{tikzcd}
$$

in which u commutes with the p_i and p_i', and $e = \varepsilon \circ u$. pi_2 corresponds to the universal property of the dependent product $f_2 = \Pi_{f_1}(g)$.

Proposition 3.7. *The model category* Lcc *is a model for the* $(2, 1)$*-category of lcc categories and lcc functors:*

(1) An object $\mathcal{C} \in$ Lcc *is fibrant if and only if its underlying category is lcc and*

- *a map $i(\mathrm{Tm}) \to U(\mathcal{C})$ is marked if and only if its image is a terminal object;*
- *a map $i(\mathrm{Pb}) \to U(\mathcal{C})$ is marked if and only if its image is a pullback square; and*
- *a map $i(\mathrm{Pi}) \to U(\mathcal{C})$ is marked if and only if its image is (isomorphic to) the diagram of the evaluation map of a dependent product.*

(2) *The homotopy category of* Lcc *is equivalent to the category of lcc categories and isomorphism classes of lcc functors.*

(3) *The homotopy function complexes of fibrant lcc sketches are given by the groupoids of lcc functors and their natural isomorphisms.*

Proof. Homotopy function complexes of maps from cofibrant to fibrant objects in a model Gpd-category can be computed as nerves of the groupoid enrichment. Thus (2) and (3) follow from (1) and Lemma 3.5.

By Hirschhorn [39, Theorem 4.1.1], the fibrant objects of the left Bousfield localization $\mathrm{Lcc} = S^{-1}\mathrm{Cat}^{\mathrm{lcc}}$ at the set S of morphisms from Definition 3.6 are precisely the fibrant lcc-marked categories $\mathcal{C}$ which are f-local for all $f \in S$. The verification of the equivalence asserted in (1) can thus be split up into three parts corresponding to terminal objects, pullback squares and dependent products. As the three proofs are very similar, we give only the proof for pullbacks. For this we must show that if $\mathcal{C}$ is a Pb-marked category, then marked maps $i(\mathrm{Pb}) \to \mathcal{C}$ are stable under isomorphisms and $\mathcal{C}$ is pb_i-local for $i = 0, 1, 2$ if and only if the underlying category $U(\mathcal{C})$ has all pullbacks and maps $i(\mathrm{Pb}) \to U(\mathcal{C})$ are marked if and only if their images are pullbacks.

Let $\mathcal{M}$ be a model Gpd-category. The homotopy function complexes of maps from cofibrant to fibrant objects in $\mathcal{M}$ can be computed as nerves of mapping groupoids. The nerve functor $N : \mathrm{Gpd} \to \mathrm{sSet}$ preserves and reflects trivial fibrations. Thus if $f : A \to B$ is a morphism of cofibrant objects A, B, then a fibrant object X is f-local if and only if

$$\mathcal{M}(f, X) : \mathcal{M}(B, X) \to \mathcal{M}(A, X)$$

is a trivial fibration of groupoids, i.e. an equivalence that is surjective on objects.

Unfolding this we obtain the following characterization of pb_i-locality for a fibrant Pb-marked category:

- $\mathcal{C}$ is pb_0-local if and only if all Pb-marked squares commute.
- $\mathcal{C}$ is pb_1-local if and only if every cospan can be completed to a Pb-marked square, and isomorphisms of cospans can be lifted uniquely to isomorphisms of Pb-marked squares completing them.

- $\mathcal{C}$ is pb_2-local if and only if every commutative square completing the lower cospan of a Pb-marked square factors via the Pb-marked square, and every factorization is compatible with natural isomorphisms of diagrams. By compatibility with the identity isomorphism, the factorization is unique.

If these conditions are satisfied, then every cospan in $\mathcal{C}$ can be completed to a pullback square which is Pb-marked, and Pb-marked squares are pullbacks. By fibrancy of $\mathcal{C}$, it follows that precisely the pullback squares are Pb-marked.

Conversely, if we take as Pb-marked squares the pullbacks in a category $\mathcal{C}$ with all pullbacks, then Pb-marked squares will be stable under isomorphism, and, by the characterization above, $\mathcal{C}$ will be pb_i-local for all i. □

3.3 Strict lcc categories

A naive interpretation of type theory in the fibrant objects of Lcc as outlined in the introduction suffers from very similar issues as Seely's original version: Type theoretic structure is preserved up to equality by substitution, but lcc functors preserve the corresponding objects with universal properties only up to isomorphism.

In this section, we explore an alternative model categorical presentation of the higher category of lcc categories. Our goal is to rectify the deficiency that lcc functors do not preserve lcc structure up to equality. Indeed, lcc structure on fibrant lcc sketches is induced by a right lifting *property*, so there is no preferred choice of lcc structure on fibrant lcc sketches. We can thus not even state the required preservation up to equality. To be able to speak of distinguished choice of lcc structure, we employ the following technical device.

Definition 3.8 (Nikolaus [66])**.** Let $\mathcal{M}$ be a combinatorial model category and let J be a set of trivial cofibrations such that objects with the right lifting property against J are fibrant. An *algebraically fibrant object* of $\mathcal{M}$ (with respect to J) consists of an object $G(X) \in \mathrm{Ob}\,\mathcal{M}$ equipped with a choice of lifts against all morphisms $j \in J$. Thus X comes with maps $\ell_X(j, a) : B \to G(X)$ for all $j : A \to B$ in J and $a : A \to G(X)$ in $\mathcal{M}$ such that

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "j"'] & G(X) \\ B \arrow[ur, "\ell_X(j,a)"'] & \end{tikzcd}$$

commutes. A morphism of algebraically fibrant objects $f : X \to Y$ is a morphism $f : G(X) \to G(Y)$ in $\mathcal{M}$ that preserves the choices of lifts, in the sense that $f \circ \ell_X(j, a) = \ell_Y(j, fa)$ for all $j : A \to B$ in J and $a : A \to G(X)$. The category of algebraically fibrant objects is denoted by $\mathrm{Alg}\,\mathcal{M}$, and the evident forgetful functor $\mathrm{Alg}\,\mathcal{M} \to \mathcal{M}$ by G.

Proposition 3.9. *Denote by* $\mathcal{I} \in \operatorname{Ob}\mathrm{Gpd}$ *the free-standing isomorphism with objects* 0 *and* 1 *and let* $K \in \operatorname{Ob} I_{\mathrm{lcc}}$. *Let* A_K, B_K *be the lcc-marked object given by* $U(A_K) = U(B_K) = \mathcal{I} \times K$ *with* $K \cong \{0\} \times K \hookrightarrow \mathcal{I} \times K$ *the only marking for* A_K *and* $K \cong \{\varepsilon\} \times K \hookrightarrow \mathcal{I} \times K$, $\varepsilon = 0, 1$ *the markings for* B_K, *and denote by* $j_K : A_K \to B_K$ *the obvious inclusion.*

Then j_K *is a trivial cofibration in* $\mathrm{Cat}^{\mathrm{lcc}}$, *and an object of* $\mathrm{Cat}^{\mathrm{lcc}}$ *is fibrant if and only if it has the right lifting property against* j_K *for all* K.

Proof. The maps j_K are injective on objects and hence cofibrations, and they reflect markings up to isomorphism, hence are also weak equivalences. A map $a : A_K \to \mathcal{C}$ corresponds to an isomorphism of maps $a_0, a_1 : i(K) \to \mathcal{C}$ with a_0 marked, and a can be lifted to B_K if and only if a_1 is also marked. Thus $\mathcal{C}$ has the right lifting property against the j_K if and only if its markings are stable under isomorphism, which is the case if and only if $\mathcal{C}$ is fibrant. □

Proposition 3.10. *An object of* Lcc *is fibrant if and only if it has the right lifting property against all of the following morphisms, all of which are trivial cofibrations in* Lcc:

(1) The maps j_K *of Proposition 3.9.*

(2) The morphisms of Definition 3.6.

(3) The maps $\langle \mathrm{id}, \mathrm{id}\rangle : B \amalg_A B \to B$, *where* $A \to B$ *is one of* $\mathrm{tm}_2, \mathrm{pb}_2$ *or* pi_2.

Proof. All three types of maps are injective on objects and hence cofibrations in $\mathrm{Cat}^{\mathrm{lcc}}$ and Lcc. By Proposition 3.9, the maps j_K are trivial cofibrations of lcc-marked categories and hence also trivial cofibrations in Lcc.

By Proposition 3.7, the fibrant objects of Lcc are precisely the lcc categories. If $\mathcal{C}$ is an lcc category and $f : X \to Y$ is a morphism of type (2) or (3), then

$$\mathrm{Lcc}(f, \mathcal{C}) : \mathrm{Lcc}(Y, \mathcal{C}) \to \mathrm{Lcc}(X, \mathcal{C})$$

is an equivalence of groupoids and hence induces a bijection of isomorphism classes. It follows by the Yoneda lemma that $\gamma(f)$ is an isomorphism in $\operatorname{Ho}\mathrm{Lcc}$, so f is a weak equivalence in Lcc.

On the other hand, let $\mathcal{C}$ be a fibrant lcc-marked category with the right lifting property against morphisms of type (2) and the morphisms of type (3). The right lifting property against $\mathrm{pb}_0, \mathrm{pb}_1$ and pb_2 implies that Pb-marked diagrams commute, that every cospan can be completed to a Pb-marked square, and that every square over a cospan factors via every Pb-marked square over the cospan. Uniqueness of factorizations follows from the right lifting property against the map of type (3) corresponding to pullbacks. Thus $\mathcal{C}$ has pullbacks, and the argument for terminal objects and dependent products is similar. □

Definition 3.11. A *strict lcc category* is an algebraically fibrant object of Lcc with respect to the set J consisting of the morphisms of types (1) – (3) of Proposition 3.10. The category of strict lcc categories is denoted by sLcc.

Remark 3.12. The objects in the image of the forgetful functor $G : \mathrm{sLcc} \to \mathrm{Lcc}$ are the fibrant lcc sketches, i.e. lcc categories. To endow an lcc category with the structure of a strict lcc category, we need to choose canonical lifts $\ell(j, -)$ against the morphisms $j \in J$. Because the lifts against all other morphisms are uniquely determined, only the choices for $\mathrm{tm}_1, \mathrm{pb}_1$ and pi_1 are relevant for this. Thus a strict lcc category is an lcc category with assigned terminal object, pullback squares and dependent products (including the evaluation maps of dependent products). A strict lcc functor is then an lcc functor that preserves these canonical choices of universal objects not just up to isomorphism but up to equality.

The slice category $\mathcal{C}_{/\sigma}$ over an object σ of an lcc category $\mathcal{C}$ is lcc again. A morphism $s : \sigma \to \tau$ in $\mathcal{C}$ induces by pullback an lcc functor $s^* : \mathcal{C}_{/\tau} \to \mathcal{C}_{/\sigma}$, and there exist functors $\Pi_s, \Sigma_s : U(\mathcal{C}_{/\tau}) \to U(\mathcal{C}_{/\sigma})$ and adjunctions $\Sigma_s \dashv U(s^*) \dashv \Pi_s$. These data depend on choices of pullback squares and dependent products, and hence they are preserved by lcc functors only up to isomorphism.

For strict lcc categories Γ, however, these functors can be constructed using canonical lcc structure, i.e. using the lifts $\ell(j, -)$ for various $j \in J$, and this choice is preserved by strict lcc functors.

Proposition 3.13. *Let Γ be a strict lcc category, and let $\sigma \in \mathrm{Ob}\,\Gamma$. Then there is a strict lcc category $\Gamma_{/\sigma}$ whose underlying category is the slice $U(G(\Gamma))_{/\sigma}$.*

If $s : \sigma \to \tau$ is a morphism in Γ, then there is a canonical choice of pullback functor $s^* : G(\Gamma_{/\tau}) \to G(\Gamma_{/\sigma})$ which is lcc (but not necessarily strict) and canonical left and right adjoints

$$\Sigma_s \dashv U(s^*) \dashv \Pi_s.$$

These data are natural in Γ. Thus if $f : \Gamma \to \Delta$ is strict lcc, then the evident functor $f_{/\sigma} : U(G(\Gamma_{/\sigma})) \to U(G(\Delta_{/f(\sigma)}))$ is strict lcc, and the following *squares in* Lcc *respectively* Cat *commute:*

$$\begin{tikzcd}
\Gamma_{/\sigma} \arrow[r, "f_{/\sigma}"] & \Delta_{/f(\sigma)} \\
\Gamma_{/\tau} \arrow[r, "f_{/\tau}"] \arrow[u, "s^*"] & \Delta_{/f(\tau)} \arrow[u, "f(s)^*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma_{/\sigma} \arrow[r, "f_{/\sigma}"] \arrow[d, "\Sigma_s"] & \Delta_{/f(\sigma)} \arrow[d, "\Sigma_{f(s)}"] \\
\Gamma_{/\tau} \arrow[r, "f_{/\tau}"] & \Delta_{/f(\tau)}
\end{tikzcd}$$

$$\begin{tikzcd}
\Gamma_{/\sigma} \arrow[r, "f_{/\sigma}"] \arrow[d, "\Pi_s"] & \Delta_{/f(\sigma)} \arrow[d, "\Pi_{f(s)}"] \\
\Gamma_{/\tau} \arrow[r, "f_{/\tau}"] & \Delta_{/f(\tau)}
\end{tikzcd}$$

(Here application of G and U has been omitted; the left square is valued in Lcc*, and the two right squares are valued in* Cat*.) $f_{/\sigma}$ and $f_{/\tau}$ commute with taking transposes along the involved adjunctions.*

Proof. We take as canonical terminal object of $\Gamma_{/\sigma}$ the identity morphism id_σ on σ. Canonical pullbacks in $\Gamma_{/\sigma}$ are computed as canonical pullbacks of the underlying diagram in Γ, and similarly for dependent products.

The canonical pullback and dependent product functors σ^*, Π_s are defined using canonical pullbacks and dependent products, and dependent sum functors Σ_s are computed by composition with s. Units and counits of the adjunctions are given by the evaluation maps of canonical dependent products and the projections of canonical pullbacks.

Because these data are defined in terms of canonical lcc structure on Γ, they are preserved by strict lcc functors. □

The context morphisms in our categories with families (cwfs) [27] will usually be defined as functors of categories in the opposite directions. Cwfs are categories equipped with contravariant functors to Fam, the category of families of sets. To avoid having to dualize twice, we thus introduce the following notion.

Definition 3.14. A *covariant cwf* is a category $\mathcal{C}$ equipped with a (covariant) functor $(\mathrm{Ty}, \mathrm{Tm}) : \mathcal{C} \to \mathrm{Fam}$.

The intuition for a context morphism $f : \Gamma \to \Delta$ in a cwf is an assignment of terms in Γ to the variables occurring in Δ. Dually, a morphism $f : \Delta \to \Gamma$ in a covariant cwf should thus be thought of as a mapping of the variables in Δ to terms in context Γ, or more conceptually as an interpretation of the mathematics internal to Δ into the mathematics internal to Γ.

Apart from our use of covariant cwfs, we adhere to standard terminology with the obvious dualization. For example, an empty context in a covariant cwf is an initial (instead of terminal) object in the underlying category.

To distinguish type and term formers in (covariant) cwfs from the corresponding categorical structure, the type theoretic notions are typeset in bold where confusion is possible. Thus $\mathbf{\Pi}_\sigma\,\tau$ denotes a dependent product type whereas $\Pi_\sigma(\tau)$ denotes application of a dependent product functor $\Pi_\sigma : \mathcal{C}_{/\sigma} \to \mathcal{C}$ to an object $\tau \in \operatorname{Ob} \mathcal{C}_{/\sigma}$.

Definition 3.15. The covariant cwf structure on sLcc is given by $\mathrm{Ty}(\Gamma) = \operatorname{Ob}\Gamma$ and $\mathrm{Tm}(\Gamma, \sigma) = \mathrm{Hom}_\Gamma(1, \sigma)$, where 1 denotes the canonical terminal object of Γ.

Proposition 3.16. *The covariant cwf* sLcc *has an empty context and context extensions, and it supports finite product and extensional equality types.*

Proof. It follows from Theorem 3.18 below that sLcc is cocomplete and that $G : \mathrm{sLcc} \to \mathrm{Lcc}$ has a left adjoint F. In particular, there exists an initial strict lcc category, i.e. an empty context in sLcc.

Let $\Gamma \vdash \sigma$. The context extension $\Gamma.\sigma$ is constructed as pushout

$$\begin{tikzcd} F(\{t, \sigma\}) \arrow[r] \arrow[d] & F(\{v : t \to \sigma\}) \arrow[d] \\ \Gamma \arrow[r, "p"] & \Gamma.\sigma \end{tikzcd}$$

where $\{t, \sigma\}$ denotes a minimally marked lcc sketch with two objects and $\{v : t \to \sigma\}$ is the minimally marked free-standing arrow. The vertical morphism on the left is induced by mapping t to 1 (the canonical terminal object of Γ) and σ to σ, and the top morphism is the evident inclusion. The variable $\Gamma.\sigma \vdash v : p(\sigma)$ is given by the image of v in $\Gamma.\sigma$.

Unit types $\Gamma \vdash 1$ are given by the canonical terminal objects of strict lcc categories Γ. Binary product types $\Gamma \vdash \sigma \times \tau$ are given by canonical pullbacks $\sigma \times_1 \tau$ over the canonical terminal object 1 in Γ. Finally, equality types $\Gamma \vdash \mathrm{Eq}\, s\, t$ are constructed as canonical pullbacks

$$\begin{tikzcd} \mathrm{Eq}\, s\, t \arrow[r] \arrow[d] & 1 \arrow[d, "t"] \\ 1 \arrow[r, "s"] & \sigma, \end{tikzcd}$$

in Γ, i.e. as equalizers of s and t.

Because these type constructors (and evident term formers) are defined from canonical lcc structure, they are stable under substitution. $\square$

Remark 3.17. Unfortunately, sLcc does not support dependent product or dependent sum types in a similarly obvious way. The introduction rule for dependent types is

$$\frac{\Gamma \vdash \sigma \qquad \Gamma.\sigma \vdash \tau}{\Gamma \vdash \mathbf{\Pi}_\sigma \tau}.$$

To interpret it, we would like to apply the dependent product functor $\Pi_\sigma : U(G(\Gamma))_{/\sigma} \to U(G(\Gamma))$ to τ.

We thus need a functor $U(G(\Gamma.\sigma)) \to U(G(\Gamma_{/\sigma}))$ to obtain an object of the slice category, and the construction of such a functor appears to be not generally possible. Note that the most natural strategy for constructing this functor using the universal property of $\Gamma.\sigma$ does not work: For this we would note that the pullback functor $\sigma^* : G(\Gamma) \to G(\Gamma_{/\sigma})$ is lcc, and that the diagonal $d : \sigma \to \sigma \times \sigma$ is a morphism $1 \to \sigma^*(\sigma))$ in $\Gamma_{/\sigma}$, and then try to obtain $\langle \sigma^*, d\rangle : \Gamma.\sigma \to \Gamma_{/\sigma}$. The flaw in this argument is that σ^*, while lcc, is not strict, and the universal property of $\Gamma.\sigma$ only applies to strict lcc functors. A solution to this problem is presented in Section 3.4.

We conclude the section with a justification for why we have not gone astray so far: The initial claim was that our interpretation of type theory would be valued in the category of lcc categories, but sLcc is neither 1-categorically nor bicategorically equivalent to the category Lcc_f of fibrant lcc sketches. Indeed, not every non-strict lcc functor of strict lcc categories is isomorphic to a strict lcc functor. Nevertheless, sLcc has model category structure that presents the same higher category of lcc categories by the following theorem:

Theorem 3.18 (Nikolaus [66] Proposition 2.4, Bourke [14] Theorem 19)**.** *Let $\mathcal{M}$ be a combinatorial model category, and let J be a set of trivial cofibrations such that objects with the right lifting property against J are fibrant. Then $G : \mathrm{Alg}\,\mathcal{M} \to \mathcal{M}$ is monadic with left adjoint F, and $\mathrm{Alg}\,\mathcal{M}$ is a locally presentable category. The model structure of $\mathcal{M}$ can be transferred along the adjunction $F \dashv G$ to $\mathrm{Alg}\,\mathcal{M}$, endowing $\mathrm{Alg}\,\mathcal{M}$ with the structure of a combinatorial model category. $F \dashv G$ is a Quillen equivalence, and the unit $X \to G(F(X))$ is a trivial cofibration for all $X \in \mathcal{M}$.*

Theorem 3.18 appears in Nikolaus [66] with the additional assumption that all cofibrations in $\mathcal{M}$ are monomorphisms. This assumption is lifted in Bourke [14], but there J is a set of generating trivial cofibrations, which is a slightly stronger condition than the one stated in the theorem. However, the proof in Bourke [14] works without change in the more general setting.

That the model structure of $\mathrm{Alg}\,\mathcal{M}$ is obtained by transfer from that of $\mathcal{M}$ means that G reflects fibrations and weak equivalences.

Lemma 3.19. *Let $\mathcal{M}$ and J be as in Theorem 3.18, and suppose furthermore that $\mathcal{M}$ is a model* Gpd*-category. Then $\mathrm{Alg}\,\mathcal{M}$ has the structure of a model* Gpd*-category, and the adjunction $F \dashv G$ lifts to a Quillen* Gpd*-adjunction.*

Proof. Let X and Y be algebraically fibrant objects. We define the mapping groupoid $(\mathrm{Alg}\,\mathcal{M})(X, Y)$ to be the full subgroupoid of $\mathcal{M}(G(X), G(Y))$ whose objects are the maps of algebraically fibrant objects $X \to Y$.

Because Gpd is generated under colimits by the free-standing isomorphism $\mathcal{I}$, it will follow from the existence of powers $X^{\mathcal{I}}$ that $\mathrm{Alg}\,\mathcal{M}$ is complete as a Gpd-category. As we will later show that G is a right adjoint, the powers in $\mathrm{Alg}\,\mathcal{M}$ must be constructed such that they commute with G, i.e. $G(X^{\mathcal{I}}) = G(X)^{\mathcal{I}}$.

Let $j : A \to B$ be in J and let $a : A \to G(X)^{\mathcal{I}}$. The canonical lift $\ell(j,a) : B \to G(X)^{\mathcal{I}}$ is constructed as follows: a corresponds to a map $\bar{a} : \mathcal{I} \to \mathcal{M}(A, G(X))$, i.e. an isomorpism of maps $A \to G(X)$. Its source and target are morphisms $\bar{a}_0, \bar{a}_1 : A \to G(X)$, for which we obtain canonical lifts $\ell(j, \bar{a}_i) : B \to G(X)$ using the canonical lifts of X. Because $G(X)$ is fibrant and j is a trivial cofibration, the map $\mathcal{M}(j, G(X)) : \mathcal{M}(B, G(X)) \to \mathcal{M}(A, G(X))$ is a trivial fibration and in particular an equivalence. It follows that $\bar{a}$ can be lifted uniquely to an isomorphism of $\ell(j, \bar{a}_0)$ with $\ell(j, \bar{a}_1)$, and we take $\ell(j,a) : B \to G(X)^{\mathcal{I}}$ as this isomorphism's transpose.

From uniqueness of the lift defining $\ell(j,a)$, it follows that a map $G(Y) \to G(X)^{\mathcal{I}}$ preserves canonical lifts if and only if the two maps

$$G(Y) \longrightarrow G(X)^{\mathcal{I}} \xrightarrow{G(X)^{\{i\}}} G(X)$$

given by evaluation at the endpoints $i = 0, 1$ of the isomorphism $\mathcal{I}$ preserve canonical lifts. Thus the canonical isomorphism

$$\mathrm{Gpd}(\mathcal{I}, \mathcal{M}(G(Y), G(X))) \cong \mathcal{M}(G(Y), G(X)^{\mathcal{I}})$$

restricts to an isomorphism

$$\mathrm{Gpd}(\mathcal{I}, (\mathrm{Alg}\,\mathcal{M})(Y, X)) \cong (\mathrm{Alg}\,\mathcal{M})(Y, X^{\mathcal{I}}).$$

It follows by Kelly and Kelly [57, theorem 4.85] and the preservation of powers by G that the 1-categorical adjunction $F \dashv G$ is groupoid enriched. It is proved in Nikolaus [66] that G, when considered as a functor of ordinary categories, is monadic using Beck's monadicity theorem. The only additional assumption for the enriched version of Beck's theorem [26, theorem II.2.1] we have to check is that the coequalizer of a G-split pair of morphisms as constructed in Nikolaus [66] is a colimit also in the enriched sense. This follows immediately from the fact that G is locally full and faithful. G is Gpd-monadic and accessible, so $\mathrm{Alg}\,\mathcal{M}$ is Gpd-cocomplete by Blackwell et al. [12, theorem 3.8].

It remains to show that $\mathrm{Alg}\,\mathcal{M}$ is groupoid enriched also in the model categorical sense. For this it suffices to note that G preserves (weighted) limits and that G preserves and reflects fibrations and weak equivalences, so that the map

$$X^{\mathcal{H}} \to X^{\mathcal{G}} \times_{Y^{\mathcal{G}}} Y^{\mathcal{H}}$$

induced by a cofibration of groupoids $f : \mathcal{G} \rightarrowtail \mathcal{H}$ and a fibration $g : X \to Y$ in $\mathrm{Alg}\,\mathcal{M}$ is a fibration and a weak equivalence if either f or g is a weak equivalence. $\square$

Remark 3.20. Lack [58] defines model category structure on T-Alg_s, the category of strict algebras and their strict morphisms for a 2-monad T on a model Gpd-category. If we choose for T the monad on Cat assigning to every category the free lcc category generated by it, then T-Alg_s is Gpd-equivalent to sLcc, so it is natural ask whether their model category structures agree.

The model category structure on T-Alg_s is defined by transfer from Cat, i.e. such that the forgetful functor T-$\mathrm{Alg}_s \to \mathrm{Cat}$ reflects fibrations and weak equivalences. The same is true for $\mathrm{sLcc} \to \mathrm{Lcc}$, and this functor is valued in the fibrant objects of Lcc. The restriction of the functor $\mathrm{Lcc} \to \mathrm{Cat}$ to fibrant objects reflects weak equivalences and trivial fibrations because equivalences of categories preserve and reflect universal objects that exist in domain and codomain. Thus sLcc and T-Alg_s have the same sets of weak equivalences and trivial fibrations, hence their model category structures coincide.

3.4 Algebraically cofibrant strict lcc categories

As noted in Remark 3.17, to interpret dependent sum and dependent product types in sLcc, we would need to relate context extensions $\Gamma.\sigma$ to slice categories $\Gamma_{/\sigma}$. In this section we discuss how this problem can be circumvented by considering yet another Quillen equivalent model category: The category of algebraically *cofibrant* strict lcc categories.

The slice category $\mathcal{C}_{/x}$ of an lcc category $\mathcal{C}$ is bifreely generated by (any choice of) the pullback functor $\sigma^* : \mathcal{C} \to \mathcal{C}_{/x}$ and the diagonal $d : x \to x \times x$, viewed as a morphism $1 \to x^*(x)$ in $\mathcal{C}_{/x}$: Given a pair of lcc functor $f : \mathcal{C} \to \mathcal{D}$ and morphism $s : 1 \to f(x)$ in $\mathcal{D}$, there is an lcc functor $g : \mathcal{C}_{/x} \to \mathcal{D}$ that commutes with f and x^* up to a natural isomorphism under which $g(d)$ corresponds to s, and every other lcc functor with this property is uniquely isomorphic to g.

Phrased in terms of model category theory, this biuniversal property amounts to asserting that the square

$$\begin{tikzcd}
\{t, x\} \arrow[r] \arrow[d] & \{d : t \to x\} \arrow[d] \\
\mathcal{C} \arrow[r, "x^*"] & \mathcal{C}_{/x}
\end{tikzcd}$$

is a homotopy pushout square in Lcc. Here $\{t, x\} = \{t, x\}^\flat$ denotes the discrete category with two objects and no markings, from which $\{d : t \to x\}$ is obtained by adjoining a single morphism $t \to x$. The left vertical map $\{t, x\} \to \mathcal{C}$ maps t to some terminal object and x to x, and the right vertical map maps d to the diagonal $x \to x^*(x)$ in $\mathcal{C}_{/x}$.

Recall from Proposition 3.16 that a context extension $\Gamma.\sigma$ in sLcc is defined by the 1-categorical pushout square

$$\begin{tikzcd} F(\{t, \sigma\}) \arrow[r] \arrow[d] & F(\{v : t \to \sigma\}) \arrow[d] \\ \Gamma \arrow[r, "p"] & \Gamma.\sigma. \end{tikzcd} \tag{3.3}$$

Because $F \dashv G$ is a Quillen equivalence, we should thus expect to find weak equivalences relating $\Gamma_{/\sigma}$ to $\Gamma.\sigma$ if the pushout (3.3) is also a *homotopy* pushout.

By Lurie [62, Proposition A.2.4.4], this is the case if Γ, $F(\{t, \sigma\})$ and $F(\{v : t, \sigma\})$ are cofibrant, and the map $F(\{t, \sigma\}) \to F(\{v : t, \sigma\})$ is a cofibration. The cofibrations of Lcc are the maps which are injective on objects. It follows that $\{t, \sigma\}$ and $\{v : t \to \sigma\}$ are cofibrant lcc sketches, and that the inclusion of the former into the latter is a cofibration. F is a left Quillen functor and hence preserves cofibrations. Thus the pushout (3.3) is a homotopy pushout if Γ is cofibrant.

Note that components of the counit $\varepsilon : FG \Rightarrow \mathrm{Id} : \mathrm{sLcc} \to \mathrm{sLcc}$ are cofibrant replacements: Every lcc sketch is cofibrant in Lcc, every strict lcc category is fibrant in sLcc, and $F \dashv G$ is a Quillen equivalence. It follows that a strict lcc category Γ is cofibrant if and only if the counit ε_Γ is a retraction, say with section $\lambda : \Gamma \to F(G(\Gamma))$.

And indeed, this section can be used to strictify the pullback functor. We have $\sigma^* : G(\Gamma) \to G(\Gamma_{/\sigma})$, which induces a strict lcc functor $\overline{\sigma^*} : F(G(\Gamma)) \to \Gamma_{/\sigma}$. Now let

$$(\sigma^*)^s : \Gamma \xrightarrow{\lambda} F(G(\Gamma)) \xrightarrow{\overline{\sigma^*}} \Gamma_{/\sigma},$$

which is naturally isomorphic to σ^*. Adjusting the domain and codomain of the diagonal d suitably to match $(\sigma^*)^s$, we thus obtain the desired comparison functor $\langle \lambda(\sigma^*)^s, d\rangle : \Gamma.\sigma \to \Gamma_{/\sigma}$.

At first we might thus attempt to restrict the category of contexts to the cofibrant strict lcc categories Γ, for which sections $\lambda : \Gamma \to F(G(\Gamma))$ exist. Indeed, cofibrant objects are stable under pushouts along cofibrations, so the context extension $\Gamma.\sigma$ will be cofibrant again if Γ is cofibrant. The dependent product type $\mathbf{\Pi}_\sigma\, \tau$ would be defined by application of

$$\Gamma.\sigma \longrightarrow \Gamma_{/\sigma} \xrightarrow{\Pi_\sigma} \Gamma$$

to τ. Unfortunately, the definition of the comparison functor $\Gamma.\sigma \to \Gamma_{/\sigma}$ required a *choice* of section $\lambda : \Gamma \to F(G(\Gamma))$, and this choice will not generally be compatible with strict lcc functors $\Gamma \to \Delta$. The dependent products defined as above will thus not be stable under substitution.

To solve this issue, we make the section λ part of the structure. Similarly to how strict lcc categories have associated structure corresponding to their fibrancy in lcc, we make the section λ witnessing the cofibrancy of strict lcc categories part of the data, and require morphisms to preserve it. We thus consider algebraically cofibrant objects, which, dually to algebraically fibrant objects, are defined as coalgebras for a cofibrant replacement comonad. As in the case of algebraically fibrant objects, we are justified in doing so because we obtain an equivalent model category:

Theorem 3.21 (Ching and Riehl [17] Lemmas 1.2 and 1.3, Theorems 1.4 and 2.5)**.** *Let $\mathcal{M}$ be a combinatorial and model* Gpd*-category. Then there are arbitrarily large cardinals λ such that*

(1) $\mathcal{M}$ is locally λ-presentable;

(2) $\mathcal{M}$ is cofibrantly generated with a set of generating cofibrations for which domains and codomains are λ-presentable objects;

(3) an object $X \in \mathrm{Ob}\,\mathcal{M}$ is λ-presentable if and only if the functor $\mathcal{M}(X, -) : \mathcal{M} \to \mathrm{Gpd}$, given by the groupoid enrichment of $\mathcal{M}$, preserves λ-filtered colimits.

Let λ be any such cardinal. Then there is a cofibrant replacement Gpd*-comonad $C : \mathcal{M} \to \mathcal{M}$ that preserves λ-filtered colimits. Let C be any such comonad and denote its category of coalgebras by $\mathrm{Coa}\,\mathcal{M}$.*

Then the forgetful functor $U : \mathrm{Coa}\,\mathcal{M} \to \mathcal{M}$ has a left adjoint V. $\mathrm{Coa}\,\mathcal{M}$ is a complete and cocomplete Gpd*-category, and $V \dashv U$ is a* Gpd*-adjunction. The model category structure of $\mathcal{M}$ can be transferred along $V \dashv U$, making $\mathrm{Coa}\,\mathcal{M}$ a model* Gpd*-category. $V \dashv U$ is a Quillen equivalence.*

For $\mathcal{M} = \mathrm{sLcc}$, the first infinite cardinal ω satisfies the three conditions of Theorem 3.21, and $C = FG$ is a suitable cofibrant replacement comonad.

Definition 3.22. The covariant cwf structure on $\mathrm{Coa}\,\mathrm{sLcc}$ is defined as the composite

$$\mathrm{Coa}\,\mathrm{sLcc} \to \mathrm{sLcc} \to \mathrm{Fam}$$

in terms of the covariant cwf structure on sLcc.

We denote by $\eta : \mathrm{Id} \Rightarrow GF : \mathrm{Lcc} \to \mathrm{Lcc}$ the unit and by $\varepsilon : FG \Rightarrow \mathrm{Id} : \mathrm{sLcc} \to \mathrm{sLcc}$ the counit of the adjunction $F \dashv G$.

Lemma 3.23. *Let $\lambda : \Gamma \to F(G(\Gamma))$ be an FG-coalgebra. Then there is a canonical natural isomorphism $\phi : G(\lambda) \cong \eta_{G(\Gamma)} : G(\Gamma) \to G(F(G(\Gamma)))$ of lcc functors which is compatible with morphisms of FG-coalgebras.*

Proof. It suffices to construct a natural isomorphism

$$\psi : \mathrm{id} \cong \eta_{G(\Gamma)} \circ G(\varepsilon_\Gamma)$$

of lcc endofunctors on $G(F(G(\Gamma)))$ for every strict lcc category Γ, because then

$$\psi \circ G(\lambda) : G(\lambda) \cong \eta_{G(\Gamma)} \circ G(\varepsilon_\Gamma \lambda) = \eta_{G(\Gamma)}$$

for every coalgebra $\lambda : \Gamma \to F(G(\Gamma))$.

$\eta_{G(\Gamma)}$ is a trivial cofibration, so the map

$$- \circ \eta_{G(\Gamma)} : \mathrm{Lcc}(G(F(G(\Gamma))), G(F(G(\Gamma)))) \to \mathrm{Lcc}(G(\Gamma), G(F(G(\Gamma)))) \quad (3.4)$$

is a trivial fibration of groupoids. By one of the triangle identities of units and counits, we have $\eta_{G(\Gamma)} \circ G(\varepsilon_\Gamma) \circ \eta_{G(\Gamma)} = \eta_{G(\Gamma)}$. Thus both $\mathrm{id}_{G(\Gamma)}$ and $\eta_{G(\Gamma)} \circ G(\varepsilon_\Gamma)$ are sent to $\eta_{G(\Gamma)}$ under the surjective equivalence (3.4), and so we can lift the identity natural isomorphism on $\eta_{G(\Gamma)}$ to an isomorphism ψ as above. Since the lift is unique, it is preserved under strict lcc functors in Γ. □

Proposition 3.24. *The covariant cwf* Coa sLcc *has an empty context and context extensions, and the forgetful functor* Coa sLcc $\to$ sLcc *preserves both.*

Proof. The model category Coa sLcc has an initial object, i.e. an empty context. Its underlying strict lcc category Γ is the initial strict lcc category, and the structure map $\lambda : \Gamma \to F(G(\Gamma))$ is the unique strict lcc functor with this signature.

Now let $\lambda : \Gamma \to F(G(\Gamma))$ be an FG-coalgebra and $\Gamma \vdash \sigma$ be a type. We must construct coalgebra structure $\lambda.\sigma : \Gamma.\sigma \to F(G(\Gamma.\sigma))$ on the context extension in sLcc such that

$$\begin{array}{ccc} \Gamma & \xrightarrow{p} & \Gamma.\sigma \\ \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \lambda.\sigma} \\ F(G(\Gamma)) & \xrightarrow{F(G(p))} & F(G(\Gamma.\sigma)) \end{array}$$

commutes, and show that the strict lcc functor $\langle f, w \rangle : \Gamma.\sigma \to \Delta$ induced by a coalgebra morphism $f : (\Gamma, \lambda) \to (\Delta, \lambda')$ and a term $\Delta \vdash w : f(\sigma)$ is a coalgebra morphism.

Let $v : 1 \to p(\sigma)$ be the variable term of the context extension of Γ by σ. Then $\eta_{\Gamma.\sigma}(v)$ is a morphism

$$\eta_{\Gamma.\sigma}(1) \to \eta_{\Gamma.\sigma}(p(\sigma)) = F(G(p))(\eta_\Gamma(\sigma))$$

in $F(G(\Gamma.\sigma))$. $\eta_{\Gamma.\sigma}(1)$ is a terminal object and hence uniquely isomorphic to the canonical terminal object 1 of $F(G(\Gamma.\sigma))$, and $F(G(p))(\eta_\Gamma(\sigma))$ is isomorphic to $F(G(p))(\lambda(\sigma))$ via a component of $F(G(p)) \circ \phi$, where ϕ is the natural isomorphism constructed in Lemma 3.23. We thus obtain a term $\Gamma.\sigma \vdash v' : F(G(p))(\lambda(\sigma))$ and can define

$$\lambda.\sigma = \langle F(G(p)) \circ \lambda, v' \rangle$$

by the universal property of $\Gamma.\sigma$. $\lambda.\sigma$ is compatible with p and λ by construction.

Now let $f : (\Gamma, \lambda) \to (\Delta, \lambda')$ be a coalgebra morphism and let $\Delta \vdash w : f(\sigma)$. We need to show that

$$\begin{tikzcd}
\Gamma.\sigma \arrow[r, "\langle f,w\rangle"] \arrow[d, "\lambda.\sigma"] & \Delta \arrow[d, "\lambda'"] \\
F(G(\Gamma.\sigma)) \arrow[r, "F(G(\langle f,w\rangle))"] & F(G(\Delta))
\end{tikzcd}$$

commutes. This follows from the universal property of $\Gamma.\sigma$: The two maps $\Gamma.\sigma \to F(G(\Delta))$ agree after precomposing $p : \Gamma \to \Gamma.\sigma$ because by assumption f is a coalgebra morphism, and they both map v to the term $F(G(\Delta)) \vdash w' : \lambda'(f(\sigma))$ obtained from w similarly to v' from v because the isomorphism ϕ constructed in Lemma 3.23 is compatible with coalgebra morphisms. □

For $\mathcal{C}$ a Gpd-category and $x \in \operatorname{Ob}\mathcal{C}$, we denote by $\mathcal{C}_{x/}$ the higher coslice Gpd-category of objects under x. Its objects are morphisms out of x, its morphisms are triangles

$$\begin{tikzcd}
& x \arrow[dl, "y_0"'] \arrow[dr, "y_1"] \arrow[d, phantom, "\overset{\phi}{\cong}"] & \\
\cdot \arrow[rr, "f"'] & {} & \cdot
\end{tikzcd}$$

in $\mathcal{C}$ which commute up to specified isomorphism ϕ, and its 2-cells $(f_0, \phi_0) \cong (f_1, \phi_1)$ are 2-cells $\psi : f_0 \cong f_1$ in $\mathcal{C}$ such that $\phi_1(\psi \circ y_0) = \phi_0$.

Definition 3.25. Let $\mathcal{C}$ be an lcc category and $x \in \operatorname{Ob}\mathcal{C}$. A *weak context extension* of $\mathcal{C}$ by x consists of an lcc functor $f : \mathcal{C} \to \mathcal{D}$ and a morphism $v : t \to f(x)$ with t a terminal object in $\mathcal{D}$ such that the following biuniversal property holds:

For every lcc category $\mathcal{E}$, lcc functor $g : \mathcal{C} \to \mathcal{E}$ and morphism $w : u \to g(x)$ in $\mathcal{E}$ with u terminal, the full subgroupoid of $\mathrm{Lcc}_{\mathcal{C}/}(f, g)$ given by pairs of lcc functor $h : \mathcal{D} \to \mathcal{E}$ and natural isomorphism $\phi : hf \cong g$ such that the square

$$\begin{tikzcd}
h(t) \arrow[r, "h(v)"] \arrow[d] & h(f(x)) \arrow[d, "\phi_x"] \\
u \arrow[r, "w"] & g(x)
\end{tikzcd}$$

in $\mathcal{D}$ commutes is contractible (i.e. equivalent to the terminal groupoid).

Remark 3.26. Note that the definition entails that mapping groupoids of lcc functors $\mathcal{D} \to \mathcal{E}$ under $\mathcal{C}$ with $\mathcal{D}$ a weak context extension are equivalent to discrete groupoids. Lcc functors $h_0, h_1 : \mathcal{D} \to \mathcal{E}$ under $\mathcal{C}$ are (necessarily uniquely) isomorphic under $\mathcal{C}$ if and only if they correspond to the same morphism $w : u \to g(x)$ in $\mathcal{E}$.

Lemma 3.27. *Let $\lambda : \Gamma \to F(G(\Gamma))$ be an FG-coalgebra and let Δ be a strict lcc category. Then the full and faithful inclusion of groupoids*

$$\mathrm{sLcc}(\Gamma, \Delta) \subseteq \mathrm{Lcc}(G(\Gamma), G(\Delta)) \tag{3.5}$$

admits a canonical retraction $f \mapsto f^s$. There is a natural isomorphism $\zeta^f : G(f^s) \cong f$, exhibiting the retract (3.5) as an equivalence of groupoids. The retraction $f \mapsto f^s$ and natural isomorphism ζ^f is Gpd*-natural in (Γ, λ) and Δ.*

Proof. Let $f : G(\Gamma) \to G(\Delta)$. The transpose of f is a strict lcc functor $\bar{f} : F(G(\Gamma)) \to \Delta$ such that $G(\bar{f})\eta = f$. We set $f^s = \bar{f}\lambda$ and $\zeta^f = G(\bar{f})\phi$ for $\phi : G(\lambda) \cong \eta$ as in Lemma 3.23. If $f = G(g)$ already arises from a strict lcc functor $g : \Gamma \to \Delta$, then $\bar{g} = g\varepsilon$ and hence $\bar{g}\lambda = g$. The action of the retraction $f \mapsto f^s$ on natural isomorphisms $f_0 \cong f_1$ is defined analogously from the Gpd-enrichment of $F \dashv G$. □

Lemma 3.28. *Let (Γ, λ) be an FG-coalgebra. Then $G(p) : G(\Gamma) \to G(\Gamma.\sigma)$ and $v : 1 \to p(\sigma)$ form a weak context extension of $G(\Gamma)$ by σ.*

Proof. Let $f : G(\Gamma) \to \mathcal{E}$ be an lcc functor and $w : t \to f(\sigma)$ be a morphism with terminal domain in $\mathcal{E}$. Let Δ be a strict lcc category such that $G(\Delta) = \mathcal{E}$. Then by Lemma 3.27 there is an isomorphism $\zeta^f : G(f^s) \cong f$ for some strict lcc functor $f^s : \Gamma \to \Delta$. Set $g = \langle f^s, w^s \rangle$, where w^s is the unique morphism in $G(\Delta)$ such that

$$\begin{tikzcd} 1 \arrow[r, "w^s"] \arrow[d] & f^s(\sigma) \arrow[d, "\zeta^f_\sigma"] \\ t \arrow[r, "w"] & f(\sigma) \end{tikzcd}$$

commutes. (Both vertical arrows are isomorphisms.) Now with $g = \langle f^s, w^s \rangle : \Gamma.\sigma \to \Delta$ we have $\zeta^f : G(g) \circ G(p) \cong f$.

Let $h : G(\Gamma.\sigma) \to \mathcal{E}$ and $\phi : h \circ G(p) \cong f$ be any other lcc functor over $G(\Gamma)$ such that

$$\begin{tikzcd} h(1) \arrow[r, "h(v)"] \arrow[d] & h(\sigma) \arrow[d, "\phi_\sigma"] \\ t \arrow[r, "w"] & f(\sigma) \end{tikzcd}$$

commutes. We need to show that h and $G(g)$ are uniquely isomorphic under $G(\Gamma)$. Lemma 3.27 reduces this to the unique existence of an extension of the isomorphism $gp \cong h^s p : \Gamma \to \Delta$ defined as composite

$$G(gp) \cong f \cong h \circ G(p) \cong G((h \circ G(p))^s) = G(h^s p)$$

to an isomorphism $g \cong h^s : \Gamma.\sigma \to \Delta$ under Γ. This follows from the construction of $\Gamma.\sigma$ as pushout

$$\begin{tikzcd}
F(\{t, \sigma\}) \arrow[r] \arrow[d] & F(\{v : t \to \sigma\}) \arrow[d] \\
\Gamma \arrow[r, "p"] & \Gamma.\sigma,
\end{tikzcd}$$

and its universal property on 2-cells. $\square$

Lemma 3.29. *Let x be an object of an lcc category $\mathcal{C}$, and let $x^* : \mathcal{C} \to \mathcal{C}_{/x}$ be any choice of pullback functor. Denote by $d = \langle \mathrm{id}_x, \mathrm{id}_x \rangle : \mathrm{id}_x \to x^*(x)$ the diagonal morphism in $\mathcal{C}_{/x}$. Then x^* and d form a weak context extension of $\mathcal{C}$ by x.*

Proof. Let $\mathcal{E}$ be an lcc category, $f : \mathcal{C} \to \mathcal{E}$ be an lcc functor and $w : t \to f(\sigma)$ be a morphism in $\mathcal{E}$ with t terminal. We define the induced lcc functor $g : \mathcal{C}_{/x} \to \mathcal{E}$ as composition

$$\mathcal{C}_{/x} \xrightarrow{f_{/x}} \mathcal{E}_{/f(x)} \xrightarrow{w^*} \mathcal{E}$$

where $w^* : \mathcal{E}_{/f(x)} \to \mathcal{E}_{/t} \xrightarrow{\sim} \mathcal{E}$ is given by a choice of pullback functor.

Let $y \in \mathrm{Ob}\,\mathcal{C}$. We denote the composite $f(x) \to t \xrightarrow{w} f(x)$ by w'. Then the two squares

$$\begin{tikzcd}
g(x^*(y)) \arrow[r] \arrow[d] & f(x \times y) \arrow[d, "f(\mathrm{pr}_1)"] \\
t \arrow[r, "w"] & f(x)
\end{tikzcd} \qquad \begin{tikzcd}
f(y) \arrow[r, "\langle w, \mathrm{id} \rangle"] \arrow[d] & f(x \times y) \arrow[d, "f(\mathrm{pr}_1)"] \\
t \arrow[r, "w"] & f(x)
\end{tikzcd}$$

are both pullbacks over the same cospan. Here $\mathrm{pr}_1 = x^*(y)$ denotes the first projection of the product defining the pullback functor x^*, and $x \times y$ is the projection's domain. (These should not be confused with canonical products in strict lcc categories; $\mathcal{C}$ and $\mathcal{D}$ are only lcc categories.) f preserves pullbacks, so $f(x \times y)$ is a product of $f(x)$ with $f(y)$. We obtain natural isomorphisms $\phi_y : g(x^*(y)) \cong f(y)$ relating the two pullbacks for all y.

The diagram

$$\begin{tikzcd}
t \arrow[r, "w"] \arrow[d, "w"] & f(x) \arrow[r] \arrow[d, "\langle w', \mathrm{id} \rangle"] & t \arrow[d, "w"] \\
f(x) \arrow[r, "f(d)"] & f(x \times x) \arrow[r, "\mathrm{pr}_1"] & f(x)
\end{tikzcd}$$

commutes, and in particular the left square commutes. It follows that ϕ is compatible with d and w.

g and ϕ are unique up to unique isomorphism because for every morphism $k : y \to x$ in $\mathcal{C}$, i.e. object of $\mathcal{C}_{/x}$, the square

$$\begin{tikzcd} k \arrow[r, "\langle k,\mathrm{id}\rangle"] \arrow[d, "k"] & x^*(y) \arrow[d, "x^*(k)"] \\ \mathrm{id}_x \arrow[r, "d"] & x^*(x) \end{tikzcd}$$

is a pullback square in $\mathcal{C}_{/x}$. □

Lemma 3.30. *Let $\lambda : \Gamma \to F(G(\Gamma))$ be an FG-coalgebra and let $\Gamma \vdash \sigma$ be a type. Then $G(p) : G(\Gamma) \to G(\Gamma.\sigma)$ and $\sigma^* : G(\Gamma) \to G(\Gamma_{/\sigma})$ are equivalent objects of the coslice category $\mathrm{Lcc}_{G(\Gamma)/}$. The equivalence $a : G(\Gamma.\sigma) \rightleftarrows G(\Gamma_{/\sigma}) : b$ can be constructed naturally in (Γ, λ) and σ, in the sense that coalgebra morphisms in (Γ, λ) preserving σ induce natural transformations of diagrams*

$$G(\Gamma.\sigma)^{\mathcal{I}} \longleftarrow G(\Gamma.\sigma) \underset{b}{\overset{a}{\rightleftarrows}} G(\Gamma_{/\sigma}) \longrightarrow G(\Gamma_{/\sigma})^{\mathcal{I}}. \tag{3.6}$$

Proof. It follows immediately from Lemmas 3.28 and 3.29 that $G(\Gamma.\sigma)$ and $G(\Gamma_{/\sigma})$ are equivalent under $G(\Gamma)$. However, a priori the corresponding diagrams (3.6) can only be assumed to vary pseudonaturally in (Γ, λ) and σ, meaning that for example the square

$$\begin{tikzcd} G(\Gamma.\sigma) \arrow[r] \arrow[d] & G(\Gamma_{/\sigma}) \arrow[d] \\ G(\Delta.f(\sigma)) \arrow[r] & G(\Delta_{/f(\sigma)}) \end{tikzcd} \tag{3.7}$$

induced by a coalgebra morphism $f : (\Gamma, \lambda) \to (\Delta, \mu)$ would only commute up to isomorphism.

The issue is that Definition 3.25 only requires that certain mapping groupoids are contractible to a point, but the choice of point is not uniquely determined. To obtain a square (3.7) that commutes up to equality, we have to explicitly construct a map $G(\Gamma.\sigma) \to G(\Gamma_{/\sigma})$ (i.e. point of the contractible mapping groupoid) and show that this choice is strictly natural.

The map $G(\Gamma.\sigma) \to G(\Gamma_{/\sigma})$ over $G(\Gamma)$ is determined up to unique isomorphism by compatibility with the (canonical) pullback functor $\sigma^* : G(\Gamma) \to G(\Gamma_{/\sigma})$ and the diagonal $d : \mathrm{id}_\sigma \to \sigma^*(\sigma)$. Recall from the proof of Lemma 3.28 that $a = \langle (\sigma^*)^s, d^s \rangle : G(\Gamma.\sigma) \to G(\Gamma_{/\sigma})$ and $\alpha = \zeta^{\sigma^*} : G(a) \circ G(p) \cong \sigma^*$ is a valid choice. d is stable under strict lcc functors, hence by Lemmas 3.13 and 3.27, a and α are natural in FG-coalgebra morphisms.

As in the proof of Lemma 3.29, the map in the other direction can be constructed as composite

$$b : G(\Gamma_{/\sigma}) \xrightarrow{p_{/\sigma}} G(\Gamma.\sigma_{/p(\sigma)}) \xrightarrow{v^*} G(\Gamma.\sigma_{/1}) \xrightarrow{\cong} G(\Gamma.\sigma),$$

where v^* is the canonical pullback along the variable v, and the components of the natural isomorphism $\beta : b\sigma^* \cong G(p)$ are the unique isomorphisms relating pullback squares

$$\begin{tikzcd} b(\sigma^*(\tau)) \arrow[r] \arrow[d] & p(\sigma) \times p(\tau) \arrow[d, "\mathrm{pr}_1"] \\ 1 \arrow[r, "v"] & p(\sigma) \end{tikzcd} \qquad \begin{tikzcd} p(\tau) \arrow[r, "\langle v, \mathrm{id} \rangle"] \arrow[d] & p(\sigma) \times p(\tau) \arrow[d, "\mathrm{pr}_1"] \\ 1 \arrow[r, "v"] & p(\sigma). \end{tikzcd}$$

All data involved in the construction are natural in Γ by Proposition 3.13, hence so are b and β.

By Remark 3.26, the natural isomorphisms $(b, \beta) \circ (a, \alpha) \cong \mathrm{id}$ and $\mathrm{id} \cong (b, \beta) \circ (a, \alpha)$ over $G(\Gamma)$ are uniquely determined given their domain and codomain. Their naturality in (Γ, λ) and σ thus follows from that of (a, α) and (b, β). □

Lemma 3.31. *Let $\lambda : \Gamma \to F(G(\Gamma))$ be an FG-coalgebra, let σ, τ be types in context Γ and let $\Gamma.\tau \vdash t : p_\tau(\sigma)$ be a term. Let $\bar{t} : \tau \to \sigma$ be the morphism in Γ that corresponds to t under the isomorphism*

$$\mathrm{Hom}_{\Gamma.\tau}(1, p_\tau(\sigma)) \cong \mathrm{Hom}_{\Gamma_{/\tau}}(\mathrm{id}_\tau, \tau^*(\sigma)) \cong \mathrm{Hom}_\Gamma(\tau, \sigma)$$

induced by the equivalence of Lemma 3.30 and the adjunction $\Sigma_\tau \dashv \tau^*$. Then the square

$$\begin{tikzcd} G(\Gamma.\sigma) \arrow[r] \arrow[d, "G(\langle p_\tau, s \rangle)"'] & G(\Gamma_{/\sigma}) \arrow[d, "\bar{t}^*"] \\ G(\Gamma.\tau) \arrow[r] & G(\Gamma_{/\tau}) \end{tikzcd}$$

in $\mathrm{Lcc}_{G(\Gamma)/}$ commutes up to a unique natural isomorphism that is compatible with FG-coalgebra morphisms in (Γ, λ).

Proof. $\bar{t}^*$ maps the diagonal of σ to the diagonal of τ up to the canonical isomorphism $\bar{t}^* \circ \sigma^* \cong \tau^*$, hence Lemma 3.28 applies. □

Theorem 3.32. *The cwf* $\mathrm{Coa\,sLcc}$ *is a model of dependent type theory with finite product, extensional equality, dependent product and dependent sum types.*

Proof. Coa sLcc has an empty context and context extensions by Proposition 3.24. Finite product and equality types are interpreted as in sLcc (see Proposition 3.16).

Let $\Gamma \vdash \sigma$ and $\Gamma.\sigma \vdash \tau$. Denote by $a : \Gamma.\sigma \to \Gamma_{/\sigma}$ the functor that is part of the equivalence established in Lemma 3.30. Then $\Gamma \vdash \mathbf{\Sigma}_\sigma\, \tau$ respectively $\Gamma \vdash \mathbf{\Pi}_\sigma\, \tau$ are defined by application of the functors

$$\Gamma.\sigma \xrightarrow{a} \Gamma_{/\sigma} \overset{\Sigma_\sigma}{\underset{\Pi_\sigma}{\rightrightarrows}} \Gamma$$

to τ.

a being an equivalence and the adjunction $\sigma^* \dashv \Pi_\sigma$ establish an isomorphism

$$\mathrm{Hom}_{\Gamma.\sigma}(1, \tau) \cong \mathrm{Hom}_{\Gamma_{/\sigma}}(\sigma^*(1), a(\tau)) \cong \mathrm{Hom}_\Gamma(1, \Pi_\sigma(a(\tau)))$$

by which we define lambda abstraction $\Gamma \vdash \lambda(t) : \mathbf{\Pi}_\sigma\, \tau$ for some term $\Gamma.\sigma \vdash t : \tau$ and the inverse to λ (i.e. application of $p_\sigma(u)$ to the variable $\Gamma.\sigma \vdash v : \sigma$ for some term $\Gamma \vdash u : \mathbf{\Pi}_\sigma\, \tau$).

Now let $\Gamma \vdash s : \sigma$ and $\Gamma \vdash t : \langle \mathrm{id}_\Gamma, s\rangle(\tau)$. The pair term $u = (s, t)$ of type $\Gamma \vdash \mathbf{\Sigma}_\sigma\, \tau$ is defined by the diagram

$$\begin{tikzcd}
\langle \mathrm{id}_\Gamma, s\rangle(\tau) \arrow[r, "\cong"] & s^*(a(\tau)) \arrow[r] \arrow[d] & \Sigma_\sigma(a(\tau)) \arrow[d, "a(\tau)"] \\
 & 1 \arrow[ul, "t"] \arrow[ur, "u"] \arrow[r, "s"] & \sigma.
\end{tikzcd}$$

Here the isomorphism $\langle \mathrm{id}, s\rangle(\tau) \cong s^*(a(\tau))$ is a component of the natural isomorphism $\langle \mathrm{id}, s\rangle \cong s^* \circ a$ constructed in Lemma 3.31, instantiated for $\tau = 1$. Given just u we recover s by composition with $a(\tau)$, and then t as composition

$$1 \xrightarrow{\langle \mathrm{id}, u\rangle} s^*(a(\tau)) \xrightarrow{\cong} \langle \mathrm{id}_\Gamma, s\rangle(\tau).$$

These constructions establish an isomorphism of terms s and t with terms u, so the β and η laws hold.

The functors $a, \sigma^*, \Sigma_\sigma, \Pi_\sigma$ and the involved adjunctions are preserved by FG-coalgebra morphisms (Proposition 3.13, Lemmas 3.30 and 3.31), so our type theoretic structure is stable under substitution. ☐

3.5 Cwf structure on individual lcc categories

In this section we show that the covariant cwf structure on Coa sLcc that we established in Theorem 3.32 can be used as a coherence method to rectify Seely's interpretation in a given lcc category $\mathcal{C}$.

Lemma 3.33. *Let $\lambda : \Gamma \to F(G(\Gamma))$ be an FG-coalgebra. Then the following categories are equivalent:*

(1) Γ^{op}*;*

(2) the category of isomorphism classes of morphisms in the restriction of the higher coslice category $\mathrm{Lcc}_{G(\Gamma)/}$ *to slice categories* $\sigma^* : G(\Gamma) \to G(\Gamma_{/\sigma})$*;*

(3) the category of isomorphism classes of morphisms in the restriction of the higher coslice category $\mathrm{Lcc}_{G(\Gamma)/}$ *to context extensions* $G(p_\sigma) : G(\Gamma) \to G(\Gamma.\sigma)$*;*

(4) the full subcategory of the 1-categorical coslice category $(\mathrm{Coa}\,\mathrm{Lcc})_{(\Gamma,\lambda)/}$ *given by the context extensions* $p_\sigma : (\Gamma, \lambda) \to (\Gamma.\sigma, \lambda.\sigma)$*.*

Proof. As noted in Remark 3.26, the higher categories in (2) and (3) are already locally equivalent to discrete groupoids and hence biequivalent to their categories of isomorphism classes.

The functor from (1) to (2) is given by assigning to a morphism $s : \tau \to \sigma$ in Γ the isomorphism class of the pullback functor $s^* : G(\Gamma_{/\sigma}) \to G(\Gamma_{/\tau})$. The isomorphism class of an lcc functor $f : G(\Gamma_{/\sigma}) \to G(\Gamma_{/\tau})$ over $G(\Gamma)$ is uniquely determined by the morphism

$$\mathrm{id}_\tau \xrightarrow{\cong} f(\mathrm{id}_\sigma) \xrightarrow{f(d)} f(\sigma^*(\sigma)) \xrightarrow{\cong} \tau^*(\sigma),$$

which in turn corresponds to a morphism $s : \tau = \Sigma_\tau \mathrm{id}_\tau \to \sigma$, and then $f \cong s^*$.

The categories (2) and (3) are equivalent because they are both categories of weak context extensions (Lemmas 3.28 and 3.29). Finally, the inclusion of (4) into (3) is an equivalence by the Lemma 3.27. Note that every strict lcc functor $\Gamma.\sigma \to \Gamma.\tau$ commuting (up to equality) with the projections p_σ and p_τ is compatible with the coalgebra structures of $\lambda.\sigma : \Gamma \to \Gamma.\sigma$ and $\lambda.\tau : \Gamma \to \Gamma.\tau$. □

Definition 3.34. Let $\mathcal{C}$ be a covariant cwf and let Γ be a context of $\mathcal{C}$. Then the *coslice covariant cwf* $\mathcal{C}_{\Gamma/}$ has as underlying category the (1-categorical) coslice category under Γ, and its types and terms are given by the composite functor $\mathcal{C}_{\Gamma/} \xrightarrow{\mathrm{cod}} \mathcal{C} \to \mathrm{Fam}$.

Lemma 3.35. *Let* $\mathcal{C}$ *be a covariant cwf and let* Γ *be a context of* $\mathcal{C}$*. Then the coslice covariant cwf* $\mathcal{C}_{\Gamma/}$ *has an initial context. If* $\mathcal{C}$ *has context extensions, then* $\mathcal{C}_{\Gamma/}$ *has context extensions, and they are preserved by* $\mathrm{cod} : \mathcal{C}_{\Gamma/} \to \mathcal{C}$*. If* $\mathcal{C}$ *supports any of finite product, extensional equality, dependent product or dependent sum types, then so does* $\mathcal{C}_{\Gamma/}$*, and they are preserved by* $\mathrm{cod} : \mathcal{C}_{\Gamma/} \to \mathcal{C}$ □

Definition 3.36. Let $\mathcal{C}$ be a covariant cwf with an empty context and context extensions. The *core* of $\mathcal{C}$ is a covariant cwf on the least full subcategory $\mathrm{Core}\,\mathcal{C} \subseteq \mathcal{C}$ that contains the empty context and is closed under context extensions, with types and terms given by $\mathrm{Core}\,\mathcal{C} \hookrightarrow \mathcal{C} \to \mathrm{Fam}$.

Lemma 3.37. *Let $\mathcal{C}$ be a covariant cwf with an empty context and context extension. If $\mathcal{C}$ supports any of finite product types, extensional equality types, dependent product or dependent sums, then so does* $\operatorname{Core}\mathcal{C}$*, and they are preserved by the inclusion* $\operatorname{Core}\mathcal{C} \hookrightarrow \mathcal{C}$.

If $\mathcal{C}$ supports unit and dependent sum types, then $\operatorname{Core}\mathcal{C}$ *is democratic, i.e. every context is isomorphic to a context obtained from the empty context by a single context extension [18].* □

Theorem 3.38. *Let $\lambda : \Gamma \to F(G(\Gamma))$ be an FG-coalgebra. Then the underlying category of* $\operatorname{Core}((\operatorname{Coa}\operatorname{sLcc})_{(\Gamma,\lambda)/})$ *is equivalent to $U(G(\Gamma))^{\mathrm{op}}$. In particular, every lcc category is equivalent to a cwf that has an empty context and context extensions, and that supports finite product, extensional equality, dependent sum and dependent product types.*

Proof. $\operatorname{Core}((\operatorname{Coa}\operatorname{sLcc})_{(\Gamma,\lambda)/})$ is a covariant cwf supporting all relevant type constructors by Lemmas 3.35 and 3.37. It is democratic and hence equivalent to category (4) of lemma 3.33.

Given an arbitrary lcc category $\mathcal{C}$, we set $\Gamma = F(\mathcal{C})$ and define coalgebra structure by $\lambda = F(\eta) : F(\mathcal{C}) \to F(G(F(\mathcal{C})))$. Then $G(\Gamma)$ is equivalent to both $\mathcal{C}$ and a cwf supporting the relevant type constructors. □

3.6 Conclusion

We have shown that the category of lcc categories is a model of extensional dependent type theory. Previously only individual lcc categories were considered as targets of interpretations. As in these previous interpretations, we have had to deal with the issue of coherence: Lcc functors (and pullback functors in particular) preserve lcc structure only up to isomorphism, whereas substitution in type theory commutes with type and term formers up to equality.

Our novel solution to the coherence problem relies on working globally, on all lcc categories at once. In contrast to some individual lcc categories, the higher category of all lcc categories is locally presentable. This allows the use of model category theory to construct a presentation of this higher category in terms of a 1-category that admits an interpretation of type theory.

While we have only studied an interpretation of a type theory with dependent sum and dependent product, extensional equality and finite product types, it is straightforward to adapt the techniques of this paper to type theories with other type constructors. For example, a dependent type theory with a type of natural numbers can be interpreted in the category of lcc categories with objects of natural numbers. Alternatively, we can add finite coproduct, quotient and list types but omit dependent products, and obtain an interpretation in the category of arithmetic universes [63, 92].

I would expect there to be a general theorem by which one can obtain a type theory and its interpretation in the category of algebras for every (higher) monad M on Cat (with the algebras of M perhaps subject to being finitely complete and stable under slicing). Such a theorem, however, is beyond the scope of the present paper.

Chapter 4

The ∞-categorical multiverse model

Abstract

Locally cartesian closed (lcc) ∞-categories are conjectured to be a semantics of intensional dependent type theory. As an extension, homotopy type theory is expected to correspond to elementary ∞-toposes. In contrast to intensional type theory and ∞-categories, the corresponding conjectures for extensional type theory and 1-categories have been resolved for some time.

Here we explore to what extent the multiverse model of Chapter 3 can be adapted to obtain a model of intensional type theory in lcc ∞-categories. To that end, we first define model categories of sketches for lex and lcc ∞-categories which are enriched over the model category of simplicial sets. We then adapt the notion of algebraically fibrant object to enriched model categories and apply it to lex and lcc sketches to obtain model categories of strict lex and lcc ∞-categories. These model categories are models of dependent type theory with weak finite limit types: Product, unit and identity types exist, but their computation rules hold only up to proposition equality.

Finally, we consider the model category of algebraically cofibrant strict lcc ∞-categories. In contrast to the 1-categorical case, this category is not closed under general context extensions. Nevertheless, we identify a subclass of *base* types, intuitively those semantic types which do not arise from type constructors, for which context extensions exist. Dependent product types $\Pi_\sigma \tau$, with propositional instead of extensional computation rules, can then be constructed as long as the domain σ is a base type.

4.1 Introduction

Extensional dependent type theory is the internal language of lcc categories, and intensional dependent type theory with function extensionality is conjectured to be an *internal language* for lcc ∞-categories. There is a similar conjecture relating Homotopy Type Theory (HoTT, intensional dependent type theory with universes and higher inductive types) to elementary ∞-toposes.

Here we explore to what extent the 1-categorical multiverse model of extensional type theory we discussed in Chapter 3 can be adapted to the ∞-categorical case. The advantage of the multiverse approach is that, while individual lcc 1-categories need not be complete or cocomplete, the category thereof is even locally presentable and can be presented by a model category. In contrast to the approaches of Kapulkin [54] and Kapulkin and Szumiło [56], this enables the use of advanced model categorical machinery such as the computation of homotopy (co)limits via resolved diagrams or algebraically (co)fibrant objects to overcome coherence problems.

Let us first recall intuitively how extensional dependent type theory can be interpreted in the category of lcc 1-categories [10]:

- Contexts Γ are interpreted as separate lcc categories.
- Types $\Gamma \vdash \sigma$ are interpreted as objects $\sigma \in \operatorname{Ob}\Gamma$.
- Terms $\Gamma \vdash s : \sigma$ are interpreted as morphisms $s : 1 \to \sigma$ in Γ whose domains are terminal objects.
- Context morphisms from Δ to Γ are interpreted as lcc functors $\Gamma \to \Delta$ in the opposite direction.
- Substitution of types and terms $\Gamma \vdash s : \sigma$ along context morphisms $f : \Gamma \to \Delta$ is interpreted as application of lcc functors to morphisms and objects.
- Context extensions $\Gamma.\sigma$ are interpreted as slice categories $\Gamma_{/\sigma}$.
- Type and term formers are interpreted by their categorical counterpart. For example, if $\Gamma \vdash \sigma_1$ and $\Gamma \vdash \sigma_2$, then the product type $\Gamma \vdash \operatorname{Prod} \sigma_1\, \sigma_2$ is interpreted as categorical product $\sigma_1 \times \sigma_2$.

This list will also serve as our guiding principle for the interpretation of intensional type theory in the category of lcc ∞-categories. Note, however, that even for 1-categories, this list can only serve as a intuitive idea for the interpretation because it suffers from two problems:

1. Type theory postulates that context extensions $\Gamma.\sigma$ satisfy an evident universal property 1-categorically, whereas the slice category $\Gamma_{/\sigma}$ satisfies the universal property bicategorically. For lcc ∞-categories Γ, the slice category $\Gamma_{/\sigma}$ satisfies the universal property only ∞-categorical.
2. An lcc category Γ is a category in which finite limits and dependent products *exist*, but there need not be a canonical choice available. The type formers of type theory are thus not well-defined. In lcc ∞-categories,

term formers are not well-defined even when the interpretation of types is fixed.

It is natural to wonder whether, assuming the axioms of choice and suitable large cardinals, problem 2 can be solved by fixing choice functions which assign to every lcc category and finite diagram a limit cone over it, and similarly for dependent products. However, one immediately runs into the problem that these choices will be preserved by lcc functors only up to isomorphism but not up to equality as would be required for an interpretation of type theory. Note also that, for 1-categories, problem 2 is only a problem for the interpretation of types as objects: Term formers correspond to morphisms which exist uniquely such that certain diagrams commute, hence once the interpretation of types is fixed, term formers are uniquely determined. This is not the case for ∞-categories, where the corresponding morphisms exist not uniquely but up to contractible homotopy.

Note that problems 1 and 2 are a mismatch of a 1-categorical property demanded by type theory and the higher categorical property which holds semantically. *Model category theory* can be understood as a method for expressing higher categorical phenomena in 1-categorical language. For example, model categories allow the computation of higher colimits via ordinary 1-categorical colimits of resolved diagrams. It is then not surprising that the solution to problems 1 and 2 presented in Bidlingmaier [10] relies crucially on model categorical machinery. Note that two model categories can present the same higher category (in this case the $(2, 1)$-category of lcc categories) despite being inequivalent as 1-categories. Since the problems arise from a mismatch of the 1-categorical and the higher categorical, it is thus possible that they occur in some model categories but not in others, even though they present the same higher category. Our strategy is thus to find a suitable model category in which 1-categorical and higher categorical constructions line up such that our coherence problems disappear. To that end, three Quillen (but not 1-categorically) equivalent model categories are constructed in Bidlingmaier [10], each a transform of the previous one and more suitable for interpreting type theory: First the model category of lcc sketches, whose subcategory of fibrant object is given by lcc categories, lcc functors and natural isomorphisms. Next the model category of strict lcc categories, which is given by the algebraically fibrant lcc sketches. Finally the algebraically cofibrant strict lcc categories.

The construction of the three model categories relies almost exclusively on general model categorical machinery that is applied iteratively to Cat, the model category of 1-categories. Just like 1-categories, ∞-categories can be organized as a model category using Joyal's model category structure on sSet. Thus the question arises whether applying the same machinery to sSet results in a solution to coherence problems and an interpretation of type theory, too. The purpose of this work is to show that this is indeed the case, but with some qualification.

First of all, 1-categories and extensional type theory are 1-truncated: Objects or types can be isomorphic in several different ways, but morphisms or terms can only be identified in at most one way, precisely when they're equal. Thus the coherence problems in the interpretation of extensional type theory arise on the object/type level only; if these are resolved, there is no problem for terms. On the other hand, ∞-categories and intensional type theory are not truncated for any n. Thus coherence problems can arise not only for equations on types, but also for equations on terms. For example, even in its intensional variant, dependent type theory has the following β-rules for products, which hold definitionally:

$$\pi_1(\mathrm{pair}\, x_1\, x_2) = x_1 \qquad\qquad \pi_2(\mathrm{pair}\, x_1\, x_2) = x_2$$

In 1-categorical semantics, these β-rules correspond to the commutativity (which is a property, not structure) of the two triangles in the diagram

$$\begin{tikzcd} & 1 \arrow[dl, bend right, "x_1"'] \arrow[d, dashed, "\langle x_1, x_2\rangle"] \arrow[dr, bend left, "x_2"] & \\ \sigma_1 & \sigma_1 \times \sigma_2 \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & \sigma_2, \end{tikzcd} \tag{4.1}$$

where 1 denotes the terminal object.

Strict lcc ∞-categories are equipped with an operation that assigns to $(2,1)$-horns, i.e. composable pairs of morphisms, canonical extensions to 2-simplices. There, the projection term $\pi_1(x_1, x_2)$ is thus interpreted as the edge $\Delta^{\{0,2\}}$ of the canonical extension to a 2-simplex of the $(2,1)$-horn given by the projection π_1 and the interpretation $\langle x_1, x_2\rangle$ of the product term:

$$\begin{tikzcd} & \sigma_1 \times \sigma_2 \arrow[dr, "\pi_1"] & \\ 1 \arrow[ur, "\langle x_1, x_2\rangle"] \arrow[rr, dashed, "\pi_1 \circ \langle x_1, x_2\rangle"] & & \sigma_1 \end{tikzcd} \tag{4.2}$$

In 1-categories, commutativity of both (4.1) and (4.2) would mean that $x_1 = \pi_1 \circ \langle x_1, x_2\rangle$, i.e. that the interpretations of x_1 and $\pi_1(x_1, x_2)$ agree. This does not hold for ∞-categories, where we are only guaranteed a homotopy between x_1 and $\pi_1 \circ \langle x_1, x_2\rangle$. Type theoretically, this means that the β-rule holds only propositionally, i.e. there is a term of type $\mathrm{Id}\, x_1\, \pi_1(x_1, x_2)$, but x_1 and $\pi_1(x_1, x_2)$ are not definitionally equal. Similar problems arise with other β- and η-rules. Thus all type constructors that are constructed here are "weak" in the sense that all equalities (except for substitution stability) hold only up to a term of the corresponding identity type.

The paper is structured along the lines of the three model categories of lcc ∞-categories we consider: In Section 4.2 we construct the model category of sketches for ∞-categories. This is entirely analogous to the 1-categorical case, except for the need for the model category to be simplicially enriched instead of groupoid enriched; the latter sufficed for lcc 1-categories because the higher category thereof is 2-truncated. A crucial aspect of the theory of lcc categories is the fact that slices of lcc categories are again lcc. We express this model categorically by showing that the cone/slice Quillen adjunction on simplicial sets extends to a Quillen adjunction on sketches for lcc ∞-categories.

In Section 4.3 we construct the model category of strict lcc ∞-categories. As in the 1-categorical case, it is defined from the model category of sketches using the formalism of algebraically fibrant objects. However, even if the base model category is simplicial, the usual construction of algebraically fibrant objects does not generally result in a simplicial model category. We show that a variant of the construction, in which lifts must be defined not only on mapping sets but on mapping spaces, does indeed result in a simplicially enriched model category, and it is this variant that we use to define the category of strict lcc ∞-categories. Similarly to the 1-categorical case, we then go on to show that our model category admits the structure of a model of type theory with identity types and finite product types, albeit only weakly so. We finish the section with a discussion of the (higher) universal property of slices of lcc ∞-categories.

Finally, in Section 4.4, we consider algebraically cofibrant strict lcc ∞-categories. Here part of the analogy with the 1-categorical case breaks down: We can only prove that context extensions, which exist unconditionally in the 1-categorical case, exist for *base* types in the ∞-categorical case. Base types are types which are not composite, i.e. not the result of a type constructor. Thus the existence of general context extensions in the category of algebraically cofibrant strict lcc 1-categories must be understood as a peculiarity of the 1-categorical world. Nevertheless, we show that the 1-categorical strictification lemma has an ∞-categorical analogue, and that dependent product types $\Pi_\sigma \tau$ can be interpreted as long as the domains σ are base types.

4.2 Sketches

In this section, we construct the model categories of sketches for lex (finitely complete) and lcc ∞-categories. Our approach follows closely that of Isaev [47], who defines a model category of ∞-categories which admit limits of a given shape. Starting from the base model category of ∞-categories, we thus consider the model category of *marked objects* in $\mathcal{M}$, i.e. simplicial sets equipped with sets of marked diagrams. The purpose of the markings is to denote diagrams which we intend to have some universal property but which might not (yet) actually satisfy them, i.e. in our case the projection maps of finite limits and evaluation maps of dependent products. We then localize the model category of marked objects at a set of morphisms which encode the axioms that marked diagrams are supposed to modify. This localization has the effect of reducing the fibrant objects to precisely those whose markings satisfy the appropriate universal property.

In addition to the machinery developed by Isaev [47], we show in the following Subsection 4.2 that model categories of marked objects have canonical simplicial structure if the base model categories are simplicial; this will be needed in later sections. We then apply the machinery in Subsection 4.2 to construct the model category of sketches for lex and lcc ∞-categories. In contrast to Isaev [47], we choose as base model category the model category sSet^+ of *(equivalence-)marked* simplicial sets (equivalently, cartesian fibrations over the point [62, Section 3.1.3]), which is simplicially enriched.

Finally, in Subsection 4.2 we show that the cone and slice adjunction on simplicial sets extends to an adjunction on sketches for lcc ∞-categories. This is a model categorical phrasing of the statement that lex and lcc categories are stable under slicing.

Marked simplicial sets and ∞-categories

Let us begin by recalling some notions of enriched model category theory and ∞-category theory. Let $\mathcal{V}$ be a cartesian closed monoidal model category. A $\mathcal{V}$-enrichment (as model category) of a model category $\mathcal{M}$ consists of a $\mathcal{V}$-enrichment of the underlying categories such that $\mathcal{M}$ is $\mathcal{V}$-bicomplete (i.e. all tensors and powers exist) and one (hence all) of the following equivalent axioms hold:

1. If $j : A \to B$ is a cofibration in $\mathcal{M}$ and $f : X \to Y$ is a fibration in $\mathcal{M}$, then the induced map
$$\mathcal{M}(B, X) \to \mathcal{M}(A, X) \times_{\mathcal{M}(A,Y)} \mathcal{M}(B, Y)$$
on hom-objects is a fibration in $\mathcal{V}$, and a trivial fibration if one of j or f are furthermore weak equivalences.

2. If $j : U \to V$ is a cofibration in $\mathcal{V}$ and $f : X \to Y$ is a fibration in $\mathcal{M}$, then the map
$$X^B \to X^A \times_{X^B} Y^B$$
induced by powering is a fibration in $\mathcal{M}$, and a trivial fibration if one of j or f are furthermore weak equivalences.

3. If $j : U \to V$ is a cofibration in $\mathcal{V}$ and $j' : A \to B$ is a cofibration in $\mathcal{M}$, then the induced map
$$j \,\square\, j' : V \otimes A \amalg_{U \times A} U \otimes B \to V \otimes B$$
induced by tensoring is a cofibration in $\mathcal{M}$, and a trivial cofibration if one of j or j' are furthermore weak equivalences.

If $F : \mathcal{V}' \rightleftarrows \mathcal{V} : G$ is a Quillen adjunction and $\mathcal{M} = \mathcal{M}_\mathcal{V}$ is $\mathcal{V}$-enriched, then the mapping objects $\mathcal{M}_{\mathcal{V}'}(X,Y) = G(\mathcal{M}_\mathcal{V}(X,Y))$ define a $\mathcal{V}'$-enrichment on the underlying model 1-category of $\mathcal{M}$. A *simplicial* model category is a model sSet-category, where sSet is endowed with the Kan model structure (see below).

The *simplex category* Δ is the category of finite linearly ordered sets. A *simplicial set* is a presheaf over Δ; the category thereof is denoted by sSet. sSet carries two model category structures of relevance to us:

1. The Kan model structure. Its cofibrations are the monomorphisms, and its fibrations are the maps with the right lifting property against the horn inclusions $\Lambda^n_i \subseteq \Delta^n, n \geq 1, 0 \leq i \leq n$. The fibrant simplicial sets are called *Kan complexes*. The weak equivalences of the Kan model structure are the *homotopy equivalences*, i.e. maps whose geometric realization is a homotopy equivalence of topological spaces. sSet is a locally finitely presentable category. The boundary inclusions $\partial\Delta^n \subseteq \Delta^n$ generate the cofibrations, and the horn inclusions $\Lambda^n_i \subseteq \Delta^n$ generate the trivial cofibrations. Consequently, sSet with the Kan model structure is combinatorial. This model structure is simplicial, i.e. enriched as a model category over itself, via its cartesian closed structure.

2. The Joyal model structure. The cofibrations are again the monomorphisms, and its fibrant objects are the ∞-*categories*, i.e. simplicial sets with the right lifting property against all inner Kan inclusions $\Lambda^n_i \subseteq \Delta^n, n \geq 2, 0 < i < n$. The weak equivalences, also referred to as *categorical equivalences*, are the maps $f : X \to Y$ such that

   $$E(\mathrm{sSet}(Y, \mathcal{C})) \to E(\mathrm{sSet}(X, \mathcal{C}))$$

   is a weak equivalence in the Kan model structure for all ∞-categories X. Here $E : \mathrm{sSet} \to \mathrm{sSet}$ denotes the functor assigning to a simplicial set the largest Kan complex contained in it. The Joyal model structure is combinatorial, but no explicit description of a generating set of trivial cofibrations is known. Consequently, no explicit description of the class of fibrations is known. Fibrations of ∞-categories are understood, however: A map $f : \mathcal{C} \to \mathcal{D}$ of ∞-categories $\mathcal{C}$ and $\mathcal{D}$ is a fibration in the Joyal model structure if and only if f is an *inner fibration* and an *isofibration*. f being an inner fibration means that it has the right lifting property with respect to the inner horn inclusions, while f being an isofibration means that for every equivalence $e : x \to y$ in $\mathcal{D}$ and every vertex y' in $\mathcal{D}$ such that $f(y') = y$, there exists an equivalence e' with codomain y' such that $f(e') = e$.

   Every categorical equivalence is a weak equivalence in the Kan model structure. Thus the Kan model structure is a localization of the Joyal model structure. In contrast to the Kan model structure, the Joyal model structure is not simplicial via the cartesian closed structure: Cartesian products and powers induce a self-enrichment over sSet with the Joyal model structure, but not over sSet with the Kan model structure.

The lack of simplicial enrichment of the Joyal model structure poses problems for us since some of our constructions are well-defined only on simplicial model categories. Fortunately, there is an alternative model category presenting the ∞-category of ∞-categories which is simplicial: The model category of cartesian fibrations over the point [62, Section 3.1.3]. The underlying category is $\mathrm{sSet}^+ = \mathrm{sSet}^+_{/\Delta^0}$, the category of marked simplicial sets. Marked simplicial sets are pairs (X, E), where X is a simplicial set and $E \subseteq X_1$ is a set of edges containing at last the degenerated edges. Morphisms $f : (X, E) \to (X', E')$ are maps $f : X \to Y$ of underlying simplicial sets such that $f(E) \subseteq E'$. The evident forgetful functor $U : \mathrm{sSet}^+ \to \mathrm{sSet}$ admits a left adjoint and a right adjoint, both of which are sections to the forgetful functor. The left adjoint $X \mapsto X^\flat$ equips simplicial sets X with their minimal marking, i.e. precisely the degenerated edges are marked. The right adjoint $X \mapsto X^\sharp$ equips simplicial sets X with their maximal marking, i.e. all edges are marked. The maximal marking functor has a further right adjoint $\mathrm{Core} : \mathrm{sSet}^+ \to \mathrm{sSet}$ which assigns to a marked simplicial set the simplicial set spanned by the marked edges. We often suppress application of the minimal marking functor, so that e.g. $\Delta^n = (\Delta^n)^\flat$ denotes the minimally marked n-simplex in positions where a marked simplicial set is expected.

sSet^+ is not a Grothendieck topos: The maps $X^\flat \to X^\sharp$ are both mono and epi, but usually not isomorphisms. sSet^+ is, however, a Grothendieck quasitopos, hence locally presentable and locally cartesian closed. Thus for marked simplicial sets X, Y, there exists a marked simplicial set $\mathrm{sSet}^+(X, Y)$ whose vertices are the maps $X \to Y$. We denote by $\mathrm{sSet}^+_\simeq(X, Y) = \mathrm{Core}(\mathrm{sSet}^+(X, Y))$ the core of the mapping objects. An n-simplex of $U(\mathrm{sSet}^+(X, Y))$ is a map $X \times (\Delta^n)^\flat \to Y$, and an edge $X \times (\Delta^1)^\flat \to Y$ is marked if it factors via $X \times (\Delta^1)^\sharp$. An n-simplex of the simplicial set $\mathrm{sSet}^+_\simeq(X, Y)$, then, is a map $X \times (\Delta^n)^\sharp \to Y$. Intuitively, we can think of $\mathrm{sSet}^+(X, Y)$ as ∞-category of functors $X \to Y$ and their (possibly non-invertible) natural transformations, whereas $\mathrm{sSet}^+_\simeq(X, Y)$ contains only the invertible natural transformations. More generally, if $\mathcal{C}$ is a category enriched over sSet^+, we denote by $\mathcal{C}_\simeq(X, Y) = \mathrm{Core}(\mathcal{C}(X, Y))$ the sSet-enrichment induced by the Core functor.

In addition to minimal and maximal marking, the forgetful functor $U : \mathrm{sSet}^+ \to \mathrm{sSet}$ has a third section. It assigns to simplicial sets X the naturally marked simplicial set $X^\natural$, in which precisely the equivalences are marked, i.e. those edges $e : x \to y$ for which there exist 2-simplices which can be depicted as

$$\begin{tikzcd} & y \arrow[dr] & \\ x \arrow[ur, "e"] \arrow[rr, "="] & & x \end{tikzcd} \qquad \begin{tikzcd} & x \arrow[dr, "e"] & \\ y \arrow[ur] \arrow[rr, "="] & & y. \end{tikzcd}$$

Let $E = \mathrm{Core} \circ \natural : \mathrm{sSet} \to \mathrm{sSet}^+ \to \mathrm{sSet}$ be the functor assigning to simplicial sets their subsets spanned by the equivalences. If $\mathcal{C}$ is an ∞-category, then

$$E(\mathrm{sSet}(K, \mathcal{C})) \cong \mathrm{sSet}^+_{\simeq}(K^\flat, \mathcal{C}^\natural)$$

for all simplicial sets K. The model category structure on sSet^+ is given as follows. The cofibrations are the monomorphisms. The fibrant objects of sSet^+ are the marked simplicial sets of the form $\mathcal{C}^\natural$ for some ∞-category $\mathcal{C}$. A map $f : X \to Y$ in sSet^+ is a weak equivalence if and only if the map

$$\mathrm{sSet}^+_{\simeq}(Y, \mathcal{C}^\natural) \to \mathrm{sSet}^+_{\simeq}(X, \mathcal{C}^\natural)$$

is a weak equivalence in the Kan model structure for all ∞-categories $\mathcal{C}$. The adjunction $\flat : \mathrm{sSet} \rightleftarrows \mathrm{sSet}^+ : U$ is a Quillen equivalence of sSet with the Joyal model category structure with sSet^+ [62, Theorem 3.1.5.1]. The adjunction $U : \mathrm{sSet}^+ \rightleftarrows \mathrm{sSet} : \sharp$ is a adjunction with the Kan model structure [62, Theorem 3.1.5.1]. If $\mathcal{C}$ is an ∞-category, then $\mathrm{Core}(\mathcal{C}^\natural)$ is a Kan complex [62, Proposition 1.2.5.3], which implies that the adjunction $\sharp : \mathrm{sSet} \rightleftarrows \mathrm{sSet}^+ : \mathrm{Core}$ is a Quillen adjunction with the Kan model structure on sSet.

It follows from Lurie [62, Corollary 3.1.4.3] that the self-enrichment via the cartesian closed structure of sSet^+ is also an enrichment in the model categorical sense. Since the Core functor is right Quillen, the change of enrichment induced by Core defines the structure of a simplicial model category on every model sSet^+-category $\mathcal{M}$. The simplicial mapping spaces are given by $\mathcal{M}_{\simeq}(X, Y) = \mathrm{Core}(\mathcal{M}(X, Y))$, tensors by $S \otimes X = S^\sharp \otimes X$ and powers by $X^S = X^{(S^\sharp)}$ for simplicial sets S and X in $\mathcal{M}$. In particular, we can regard sSet^+ itself as a simplicial model category. Since $\sharp$ is a right Quillen functor and a section to Core, we can regard simplicial model categories as a particular case of model sSet^+-categories for which the marked simplicial sets $\mathcal{M}(X, Y)$ of maps $X \to Y$ are valued in maximally marked simplicial sets.

Note that the forgetful functor $U : \mathrm{sSet}^+ \to \mathrm{sSet}$ preserves fibrations and trivial fibration, and that it preserves and reflects cofibrations. The natural marking functor $(-)^\natural$ and U restrict to an isomorphism of subcategories of fibrant objects, and both $(-)^\natural$ and U preserve weak equivalences . By the following lemma, with a proof due to Alexander Campbell[1], U also preserves and reflects fibrations on fibrant objects:

Lemma 4.1. *Let $f : \mathcal{C} \to \mathcal{D}$ be a map of fibrant objects $\mathcal{C}, \mathcal{D}$ in* sSet^+. *Then f is a fibration if and only if the underlying map of simplicial sets is a fibration in the Joyal model structure.*

Proof. Factor f as

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "f"] \arrow[dr, "j"'] & & \mathcal{D} \\ & \mathcal{E} \arrow[ur, "p"'] & \end{tikzcd}$$

[1] `https://mathoverflow.net/a/404540`

with j a trivial cofibration and p a fibration in sSet^+. Since $\mathcal{D}$ is fibrant and p is a fibration, it follows that $\mathcal{E}$ is fibrant. Because U preserves cofibrations and preserves weak equivalences of fibrant objects, $U(f)$ is a trivial cofibration. Since U is a right Quillen functor, $U(p)$ is a fibration. It follows that the fibration $U(f)$ is a retract of $U(p)$. Because U is full and faithful on fibrant objects, it follows that f is a retract of p. By construction, p is a fibration, hence so is f. $\square$

Recall that for (unmarked) simplicial sets K, L, there exist categorically equivalent join constructions $K \star L$ (the ordinary join) and $K \diamond L$ (the alternative join). This categorical equivalence is given by a quotient map $\phi : K \diamond L \to K\star$ as in Lurie [62, Proposition 4.2.1.2] which is uniquely determined by commutativity of the following diagram:

$$\begin{tikzcd}
 & K \amalg L \arrow[dl] \arrow[dr] & \\
K \diamond L \arrow[rr, "f"] \arrow[dr] & & K \star L \arrow[dl] \\
 & \Delta^0 \diamond \Delta^0 = \Delta^1 = \Delta^0 \star \Delta^0 &
\end{tikzcd}$$

By Riehl and Verity [72, Lemma 2.4.12], the functors $\mathrm{sSet} \to \mathrm{sSet}_{K/}$ given by the (alternative) joins $K \star -, K \diamond -, - \star K, - \diamond K$ for fixed K are left Quillen functors from the Joyal model structure to the coslice model category over K. Their right adjoints are given by the (alternative) slices $L_{/p}, L^{/p}$ and coslices $L_{p/}, L^{p/}$ for maps $p : K \to L$. Transposing the comparison maps $\phi : K \diamond L \to K \star L$, we obtain maps $L_{/p} \to L^{/p}$ and $L_{p/} \to L^{p/}$, and these, too, are weak equivalences whenever L is an ∞-category.

We now lift both join operations and their parametric right adjoints to marked simplicial sets such that they commute with the forgetful functor $U : \mathrm{sSet}^+ \to \mathrm{sSet}$. Let K and L be marked simplicial sets. The marked join $K \star L$ is given by the join $U(K) \star U(L)$ with the least markings such that $U(K) \to U(K) \star U(L)$ and $U(L) \to U(K) \star U(L)$ preserve markings. The marked alternative join $K \diamond L$ is defined similarly.

Let $p : K \to L$ be a map of marked simplicial sets. Then the marked slice $K_{/p}$ is given by the slice $U(K)_{/U(p)}$, and an edge $\Delta^1 \to U(K_{/p})$ is marked if and only if its image in $U(K)$ is marked. The coslice $K_{p/}$ is defined similarly, as are alternative slice $K^{/p}$ and alternative coslice $K^{p/}$. One can then show that the marked (co)slice functors so-defined are indeed right adjoints to the marked join functors.

Note that even for unmarked simplicial sets, the ordinary join and (co)slice functors are only 1-categorically adjoint. The alternative join and alternative (coslice) adjunctions on sSet, on the other hand, are simplicial with respect to the simplicial enrichment via cartesian closure. This holds because the squares

$$
\begin{tikzcd}
S \times L \arrow[r] \arrow[d] & L \arrow[d] & S \times K \arrow[r] \arrow[d] & K \arrow[d] \\
S \times (K \diamond L) \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & (S \times K) \diamond L & S \times (K \diamond L) \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & K \diamond (S \times L)
\end{tikzcd}
\tag{4.3}
$$

are pushout squares for all simplicial sets S, K and L [70, 51.16]. For the alternative slice, the natural isomorphism $\mathrm{sSet}_{L/}(K \diamond L, P) \cong \mathrm{sSet}(X, P^{/p})$ for simplicial sets K, L and maps $p : L \to P$ is then induced by the Yoneda lemma from

$$
\begin{aligned}
&\mathrm{Hom}(S, \mathrm{sSet}_{L/}(K \diamond L, P)) \\
\cong\ &\mathrm{Hom}_{S\times L/}(S \times (K \diamond L), P) \\
\cong\ &\mathrm{Hom}_{L/}((S \times K) \diamond L, P) \\
\cong\ &\mathrm{Hom}(S \times K, P^{p/}) \\
\cong\ &\mathrm{Hom}(S, \mathrm{sSet}(K, P^{p/}))
\end{aligned}
$$

for all simplicial sets S. Here in the second line we consider P as an object under $S \times L$ via the map $S \times L \to L \xrightarrow{p} P$. The analogous isomorphism for the alternative coslice is defined similarly using the other pushout square of (4.3).

It can be shown that the squares (4.3) are also pushout squares when S, K and L are marked simplicial sets. It follows that marked alternative join and marked alternative (co)slices are adjoint in the sSet^+-enriched and sSet-enriched sense. Model categorically, marked joins and slices behave much like their unmarked versions. To prove this, we need the following lemma:

Lemma 4.2. *Let $f : K \to L$ be a map of marked simplicial sets which generates markings, i.e. edges in $x : \Delta^1 \to U(L)$ are marked if and only if there exists a marked edge $x' : \Delta^1 \to U(K)$ such that $x = fx'$. Suppose that $U(f)$ is a categorical equivalence. Then f is a weak equivalence in* sSet^+.

Proof. Let $\mathcal{C}$ be an ∞-category. Since f generates markings, a map $g : U(L) \to U(\mathcal{C}^\natural)$ preserve preserve markings if and only if $gf : U(K) \to U(L) \to U(\mathcal{C}^\natural)$ preserves markings. Thus

$$
\begin{tikzcd}
\mathrm{sSet}^+_{\simeq}(L, \mathcal{C}^\natural) \arrow[r] \arrow[d] & \mathrm{sSet}^+_{\simeq}(U(L)^\flat, \mathcal{C}^\natural) \arrow[d] \\
\mathrm{sSet}^+_{\simeq}(K, \mathcal{C}^\natural) \arrow[r] & \mathrm{sSet}^+_{\simeq}(U(K)^\flat, \mathcal{C}^\natural)
\end{tikzcd}
$$

is a pullback square. The bottom arrow is a fibration because $U(K)^\flat \to K$ is a monomorphism and hence a cofibration in sSet^+. The right vertical arrow is isomorphic to the map $E(\mathrm{sSet}(U(L), \mathcal{C})) \to E(\mathrm{sSet}(U(K), \mathcal{C}))$, which is a homotopy equivalence of simplicial sets because $U(f)$ is a categorical equivalence. It follows by right properness of the Kan model structure that also the left vertical map is a homotopy equivalence. Letting $\mathcal{C}$ vary, we conclude that f is a weak equivalence in sSet^+. $\square$

Proposition 4.3. *Let K and L be marked simplicial sets. Then the comparison map $K \diamond L \to K \star L$ of marked joins is a weak equivalence in* sSet^+. *If $\mathcal{C}$ is an ∞-category and $p : K \to \mathcal{C}^\natural$ is a map of marked simplicial sets, then the comparison maps $(\mathcal{C}^\natural)_{/p} \to (\mathcal{C}^\natural)^{/p}$ and $(\mathcal{C}^\natural)_{p/} \to (\mathcal{C}^\natural)^{p/}$ are weak equivalences in* sSet^+.

Proof. By Lemma 4.2. $\square$

Lemma 4.4. *Let $f : K \to L$ be a Kan fibration of simplicial sets K, L. If there exists a family of maps $(\ell_i : L_i \to L)_{i \in I}$ such that every vertex of L is in the image of ℓ_i for some i and the pullback $\ell_i^*(f)$ is a homotopy equivalence for all i, then f is a trivial Kan fibration.*

Proof. The Kan fibration f is a trivial Kan fibration if and only if the pullback of f along every vertex $x : \Delta^0 \to L$ in the codomain is a trivial Kan complex. By assumption, x factors via L_i for some $i \in I$. We thus have a diagram

$$\begin{tikzcd}
f^{-1}(\{x\}) \arrow[r] \arrow[d] & L_i \times_L K \arrow[r] \arrow[d, "k_i^*(f)"] & K \arrow[d, "f"] \\
\Delta^0 \arrow[r] & L_i \arrow[r, "k_i"] & L
\end{tikzcd}$$

of pullback squares. Fibrations and trivial fibrations are stable under pullback, hence $k_i^*(f)$ is a trivial fibration, and $f^{-1}(\{x\})$ is a trivial Kan complex. $\square$

Next we show that, as in the unmarked case, the marked join/(co)slice adjunction are Quillen.

Proposition 4.5. *Let K be a marked simplicial set. Then all of*

$$K \star - \qquad - \star K \qquad K \diamond - \qquad - \diamond K$$

are left Quillen functors $\mathrm{sSet}^+ \to \mathrm{sSet}^+_{K/}$.

Proof. The join and alternative join functors on simplicial sets preserve cofibrations (i.e. monomorphisms) in each argument. Since the forgetful functor $\mathrm{sSet}^+ \to \mathrm{sSet}$ reflects cofibrations, it follows that the marked join and marked alternative join functors preserve cofibrations in each argument.

Thus it remains to show that trivial cofibrations $f : K \to K'$ and $g : L \to L'$ of marked simplicial sets are preserved. By Proposition 4.3, there is a commutative diagram

$$\begin{tikzcd} K \diamond L \arrow[r] \arrow[d, "f\diamond g"'] & K \star L \arrow[d, "f\star g"] \\ K' \diamond L' \arrow[r] & K' \star L' \end{tikzcd}$$

in which both horizontal maps are weak equivalences. By two-out-of-three for weak equivalences, $f \star g$ is a trivial cofibration if and only if $f \diamond g$ is a trivial cofibration. Thus it suffices to show that $f \diamond g$ is a trivial cofibration. Since $f \diamond g = (g \diamond \mathrm{id}) \circ (\mathrm{id} \diamond f)$ and the case $f = \mathrm{id}$ is dual to $g = \mathrm{id}$, we further reduce to $g = \mathrm{id}$.

Let $\mathcal{C}$ be a ∞-category. We need to show that the Kan fibration $\bar{f}$: $\mathrm{sSet}^+_\simeq(K' \diamond L, \mathcal{C}^\natural) \to \mathrm{sSet}^+_\simeq(K \diamond L, \mathcal{C}^\natural)$ is a trivial Kan fibration. Consider for some map $p : L \to \mathcal{C}^\natural$, i.e. vertex $p : \Delta^0 \to \mathrm{sSet}^+_\simeq(L, \mathcal{C}^\natural)$, the diagram

$$\begin{tikzcd} \mathrm{sSet}^+_\simeq(K', (\mathcal{C}^\natural)^{/p}) \arrow[r] \arrow[d, "\bar{f}_p"'] & \mathrm{sSet}^+_\simeq(K' \diamond L, \mathcal{C}^\natural) \arrow[d, "\bar{f}"] \\ \mathrm{sSet}^+_\simeq(K, (\mathcal{C}^\natural)^{/p}) \arrow[r, "\bar{p}"] \arrow[d] & \mathrm{sSet}^+_\simeq(K \diamond L, \mathcal{C}^\natural) \arrow[d] \\ \Delta^0 \arrow[r, "p"] & \mathrm{sSet}^+_\simeq(L, \mathcal{C}^\natural). \end{tikzcd}$$

It follows from the enrichment of the marked alternative join/slice adjunction that the lower square and the outer rectangle are pullback squares, hence by the pasting law also the upper square is a pullback. The map $\mathcal{C}^{/U(p)} \to \mathcal{C}$ reflects equivalences, hence $(\mathcal{C}^\natural)^{/p} = (\mathcal{C}^{/U(p)})^\natural$ is a naturally marked ∞-category. f is assumed to be a trivial cofibration, so $\bar{f}_p$ is a trivial Kan fibration. Since every map $K \diamond L \to \mathcal{C}^\natural$ can be restricted to a map $p : L \to \mathcal{C}^\natural$, the family of maps $\bar{p}$ for all p satisfies the assumptions of Lemma 4.4. Thus $\bar{f}$ is indeed a trivial Kan fibration. $\square$

In the remainder of the paper, we will almost exclusively work with the model category sSet^+ instead of sSet and the Joyal model structure due to the simplicial enrichment of sSet^+. Thus "∞-category" usually refers to a naturally marked simplicial set with the right lifting property against inner horn inclusions. Note that the minimal marking functor $(-)^\flat$ (whose application we usually suppress) preserves cofibrations and trivial cofibrations. Since most trivial cofibration in sSet^+ constructed in this paper are minimally marked maps of simplicial sets, it is usually sufficient to verify that the map is a trivial cofibration in the Joyal model structure. Similarly, most fibrations we consider are maps of ∞-categories. Thus to show that the maps in question are indeed fibrations it suffices to check that the underlying map in sSet is an inner fibration and an isofibration.

Model categories of marked objects

Here we show that model categories of marked objects [47] are simplicial if the underlying model category is simplicial, and sSet$^+$-enriched if the underlying model category is sSet$^+$-enriched. Recall that a marked object in a model category $\mathcal{M}$ with respect to a *shape functor* $i : I \to \mathcal{M}$ valued in cofibrant objects consists of an underlying object $U(X)$ in $\mathcal{M}$ equipped with a set of morphisms of the form $k : i(K) \to U(X)$, the *markings* which is stable under precomposition by images of morphisms in I. We denote the category of marked objects by $\mathcal{M}^i$. The forgetful functor $U : \mathcal{M}^i \to \mathcal{M}$ has both a left adjoint $(-)^\flat$ given by endowing and a right adjoint $(-)^\sharp$. If X is an object in $\mathcal{M}$, then $X^\flat$ is such that no map $i(K) \to U(X^\flat)$ is marked, while for $X^\sharp$ every map $i(K) \to U(X^\sharp)$ is marked. $\mathcal{M}^i$ carries the structure of a model category such that $U : \mathcal{M}^i \to \mathcal{M}$ preserves and reflects cofibrations, the fibrant objects of $\mathcal{M}^i$ are those whose image in $\mathcal{M}$ is fibrant and in which the markings are stable under homotopy, and the weak equivalences are those whose image in $\mathcal{M}$ is a weak equivalence and which reflect markings up to homotopy. Thus a map $f : X \to Y$ in $\mathcal{M}^i$ is a weak equivalence if and only if the induced map $\gamma(f) : \gamma(X) \to \gamma(Y)$ is an isomorphism in $\mathrm{Ho}(\mathcal{M})^{\gamma i}$, i.e. and isomorphism of marked objects in the homotopy category with respect to the shape functor $I \xrightarrow{i} \mathcal{M} \xrightarrow{\gamma} \mathrm{Ho}(\mathcal{M})$.

Proposition 4.6. *Let $\mathcal{M}$ be a model* sSet$^+$*-category in which every object is cofibrant and let $i : I \to \mathcal{C}$ be a diagram in $\mathcal{C}$. Then the model category $\mathcal{M}^i$ carries the structure of a model* sSet$^+$*-category such that the mapping objects $\mathcal{M}^i(X, Y) \subseteq \mathcal{M}(U(X), U(Y))$ are given by the full marked simplicial subsets spanned by the vertices corresponding to marking preserving maps $X \to Y$. If $\mathcal{M}$ is simplicially enriched (i.e. if $\mathcal{M}(X, Y)$ is maximally marked for all X, Y), then also $\mathcal{M}^i$ is simplicially enriched. The adjunctions $(-)^\flat \dashv U \dashv (-)^\sharp$ extend to* sSet$^+$*-adjunctions.*

Proof. Let us first show that $\mathcal{M}^i$ is complete and cocomplete as a sSet$^+$-enriched category, i.e. show that all tensors and powers exist. Let S be a marked simplicial set and let X be a marked object. We define the tensor $S \otimes X$ by $U(S \otimes X) = S \otimes U(X)$ such that maps $k : i(K) \to S \otimes U(X)$ are marked if and only if they are of the form

$$k : i(K) \xrightarrow{k_0} U(X) \cong \Delta^0 \otimes U(X) \xrightarrow{s \otimes \mathrm{id}} S \otimes U(X)$$

for some marked $k_0 : i(K) \to U(X)$ and vertex $s : \Delta^0 \to S$. Powers are defined dually, and the universal properties hold by construction.

Next let us verify the pushout product axioms, i.e. that for all maps $i : S \to T$ of marked simplicial sets and maps $j : X \to Y$ of marked objects, the canonical map $i \,\square\, j : S \otimes Y \amalg_{S \otimes X} T \otimes X \to T \otimes Y$

1. is a cofibration if both i and j are cofibrations, and that

2. it is a trivial cofibration if furthermore one of i or j is a trivial cofibration.

Since U reflects cofibrations, 1 follows from the pushout product axiom in $\mathcal{M}$. For 2 we have to show in both cases that $i \square j$ reflects markings up to homotopy. Every marking in $T \otimes Y$ is of the form

$$K = \Delta^0 \otimes K \xrightarrow{t \otimes k} T \otimes U(Y)$$

for some marked map $k : K \to U(Y)$ and vertex $t : \Delta^0 \to T$.

Suppose first that i is a trivial cofibration of marked simplicial sets. Note that the fibrant marked simplicial sets are those with the right lifting property against a set of trivial cofibrations which are isomorphisms on vertices. It follows by the small object argument, that there exists a fibrant replacement functor $\eta : \mathrm{Id} \Rightarrow R : \mathrm{sSet}^+ \to \mathrm{sSet}^+$ such that η_K is a trivial cofibration for all marked simplicial sets K and an isomorphism on vertices. Then $R(i) : R(S) \to R(T)$ is a weak equivalence of fibrant objects, hence $\mathrm{Core}(R(I)) : \mathrm{Core}(R(S)) \to \mathrm{Core}(R(T))$ is a homotopy equivalence of Kan complexes. Thus there exists an edge $e : (\Delta^1)^\sharp \to R(T)$ such that $e(\Delta^{\{1\}}) = t$ and $e(\Delta^{\{0\}}) = R(i)(\eta_S(s))$ for some $s : \Delta^0 \to S$. We claim that

$$K \xrightarrow{s \otimes k} S \otimes U(Y) \to S \otimes U(Y) \amalg_{S \otimes U(X)} T \otimes U(Y)$$

is a preimage up to homotopy of $t \otimes k$ under $i \square j$. Let $\eta' : \mathrm{Id} \Rightarrow R' : \mathcal{M} \to \mathcal{M}$ be a fibrant replacement functor. A left homotopy h relating the two maps

$$K \rightrightarrows T \otimes U(Y) \to R'(T \otimes U(Y))$$

is given as composite

$$(\Delta^1)^\sharp \otimes K \xrightarrow{e \otimes K} R(T) \otimes K \xrightarrow{c} R'(T \otimes U(Y)).$$

Here the comparison map c is constructed as follows. The universal property of the tensor $T \otimes U(Y)$ is given by a map $T \to \mathcal{M}(U(Y), T \otimes U(Y))$. The maps $k : i(K) \to U(Y)$ and $T \otimes U(Y) \to R'(T \otimes U(Y))$ then induce a map $T \to \mathcal{M}(i(K), R'(T \otimes U(Y)))$, and the codomain of this map is a fibrant marked simplicial set because $i(K)$ is cofibrant and R' is a fibrant replacement functor. Since $T \to R(T)$ is a trivial cofibration, we obtain a map $R(T) \to \mathcal{M}(i(K), R'(T \otimes U(Y)))$, which corresponds to the map $c : R(T) \otimes i(K) \to R'(T \otimes U(Y))$.

Now suppose that j is a trivial cofibration of marked objects. The pushout product $i \square j$ is defined by a diagram

$$\begin{tikzcd}
S \otimes X \arrow[r] \arrow[d, "S \otimes j"'] & T \otimes X \arrow[d, "f"] \arrow[ddr, bend left, "T \otimes j"] & \\
S \otimes Y \arrow[r] \arrow[drr, bend right] & \cdot \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "i \square j"] & \\
& & T \otimes Y.
\end{tikzcd}$$

If $S \otimes j$ is a trivial cofibration, then f is a trivial cofibration, and if furthermore $B \otimes j$ is a trivial cofibration, then $i \Box j$ is a weak equivalence by 2-out-of-3. Thus it suffices to show that $T \otimes j$ is a trivial cofibration for all marked simplicial sets T. Since every marked simplicial set is cofibrant, $T \otimes - : \mathcal{M} \to \mathcal{M}$ is a left Quillen functor, and $U : \mathcal{M}^i \to \mathcal{M}$ reflects cofibrations. Thus it only remains to show that the marking $T \otimes j$ reflects some marking $i(K) \xrightarrow{t \otimes k} T \otimes U(Y)$ up to homotopy if j is a weak equivalence in $\mathcal{M}^i$. Since j is a weak equivalence, k has a preimage $k' : i(K) \to U(X)$ up to a homotopy h forming a commuting diagram

$$\begin{tikzcd}
i(K) \arrow[rr, bend left, "\Delta^{\{0\}}\otimes i(K)"] \arrow[d, "k'"'] & i(K) \arrow[r, "\Delta^{\{1\}}\otimes i(K)"] \arrow[d, "k"] & (\Delta^1)^\sharp \otimes i(K) \arrow[ddl, bend left, "h"] \\
U(X) \arrow[r, "U(j)"] \arrow[d, "\eta'_{U(X)}"'] & U(Y) \arrow[d, "\eta'_{U(Y)}"] & \\
R'(U(X)) \arrow[r, "R'(U(j))"] & R'(U(Y)). &
\end{tikzcd}$$

Here $\eta' : \mathrm{Id} \Rightarrow R' : \mathcal{M} \to \mathcal{M}$ denotes the unit of a fibrant replacement functor with trivial cofibrations as components. Then

$$\begin{tikzcd}
i(K) \arrow[rr, bend left, "\Delta^{\{0\}}\otimes i(K)"] \arrow[d, "t\otimes k'"'] & i(K) \arrow[r, "\Delta^{\{1\}}\otimes i(K)"] \arrow[d, "t\otimes k"] & (\Delta^1)^\sharp \otimes i(K) \arrow[ddl, bend left, "t\otimes h"] \\
T \otimes U(X) \arrow[r, "U(j)"] \arrow[d, "T\otimes\eta'_{U(X)}"'] & T \otimes U(Y) \arrow[d, "T\otimes\eta'_{U(Y)}"] & \\
T \otimes R'(U(X)) \arrow[r, "R'(U(j))"] & T \otimes R(U(Y)) &
\end{tikzcd}$$

commutes, and the maps $T \otimes \eta_{U(X)}$ and $T \otimes \eta_{U(Y)}$ are weak equivalences because $T \otimes - : \mathcal{M} \to \mathcal{M}$ preserves trivial cofibrations. It follows that $t \otimes k'$ is a preimage of $s \otimes k'$ up to homotopy. $\square$

Proposition 4.7. *Let $i : I \to \mathcal{M}$ be a functor from a small category I to a model category $\mathcal{M}$ and let $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ be an adjunction with a category $\mathcal{N}$.*

1. *The adjunction $F \dashv G$ induces an adjunction $F^i : \mathcal{M}^i \rightleftarrows \mathcal{N}^{Fi}$ of categories of marked objects.*

2. *If $\mathcal{M}$ and $\mathcal{N}$ are* sSet$^+$*-enriched, then $F^i \dashv G^i$ is* sSet$^+$*-enriched.*

3. *If $\mathcal{M}$ and $\mathcal{N}$ are model categories, i is valued in the cofibrant objects of $\mathcal{M}$ and $F \dashv G$ is a Quillen adjunction, then $F^i \dashv G^i$ is a Quillen adjunction. If $F \dashv G$ is a Quillen equivalence, then $F^i \dashv G^i$ is a Quillen equivalence.*

Proof. 1. The left adjoint F^i assigns to an object X in $\mathcal{M}^i$ the object $F(U(X))$ in $\mathcal{M}$ whose markings are maps of the form $F(i(K)) \xrightarrow{F(k)} F(U(X))$ for marked maps $k : i(K) \to U(X)$ in X. The right adjoint G^i assigns to marked objects Y in $\mathcal{N}^i$ the underlying object $G(U(Y))$ whose markings are given by the transposes $\bar{k} : i(K) \to G(U(Y))$ of marked maps $k : F(i(K)) \to U(Y)$.

2. This follows from F^i and G^i being 1-categorical adjoint because the marked simplicial sets $\mathcal{M}^i(X, Y)$ are given by the full subsets of $\mathcal{M}(U(X), U(Y))$ of spanned by the vertices corresponding to marking preserving maps, and similarly for $\mathcal{N}$.

3. Note first that $\mathcal{N}^{Fi}$ is a well-defined model category because the left Quillen functor F preserves cofibrant objects. Clearly F^i preserves cofibrations. Preservation of weak equivalences follows from the square

$$\begin{tikzcd} \mathcal{M}^i \arrow[r, "F^i"] \arrow[d, "\gamma^i"'] & \mathcal{N}^{Fi} \arrow[d, "\gamma^{Fi}"] \\ \mathrm{Ho}(\mathcal{M})^{\gamma i} \arrow[r, "\mathrm{Ho}(F)^{\gamma i}"] & \mathrm{Ho}(\mathcal{N})^{\gamma Fi}, \end{tikzcd}$$

which commutes up to natural isomorphism: The weak equivalences in the two model categories $\mathcal{M}^i$ and $\mathcal{N}^{Fi}$ are those whose morphisms whose image in $\mathrm{Ho}(\mathcal{M})^{\gamma i}$ and $\mathrm{Ho}(\mathcal{N})^{\gamma Fi}$ are isomorphisms, respectively. The same diagram implies that Quillen equivalences $F \dashv G$ induce Quillen equivalences $F^i \dashv G^i$. □

Model categories of lex and lcc ∞-categories

Here we use the machinery of marked objects to define model categories of sketches for finitely complete (lex) and locally cartesian closed (lcc) ∞-categories. Intuitively, sketches should be thought of as blueprints for the actual objects we are interested in.

Definition 4.8. Let $\mathcal{C}$ be a finitely complete ∞-category and consider a diagram $z \xrightarrow{g} x \xrightarrow{f} y$ in $\mathcal{C}$. Denote by $(\mathcal{C}^{\Delta^1})^{\mathrm{pb}}_{/f}$ the full subcategory of the slice over f spanned by pullback squares:

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \cdot \arrow[d] \\ x \arrow[r, "f"] & y \end{tikzcd}$$

A *dependent product f and g* is a terminal object in the ∞-category $\mathcal{C}_{f,g}$ determined by the following pullback square:

$$\begin{tikzcd}
\mathcal{C}_{f,g} \arrow[r] \arrow[d] & (\mathcal{C}^{\Delta^1})^{\mathrm{pb}}_{/f} \arrow[d] \\
\mathcal{C}_{/g} \arrow[r] & \mathcal{C}_{/x}.
\end{tikzcd} \tag{4.4}$$

Here the right vertical arrow is induced by the domain functor $\mathcal{C}^{\Delta^1} \to \mathcal{C}$.

Thus a dependent product of f and g is terminal among the diagrams of the following shape:

$$\begin{tikzcd}
\cdot & \cdot \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \cdot \arrow[d] \\
 & \cdot \arrow[r, "f"'] \arrow[from=ul, "g"'] & \cdot
\end{tikzcd}$$

Note that if $\mathcal{C}$ has all finite limits, then the functor $(\mathcal{C}^{\Delta^1})^{\mathrm{pb}}_{/f} \to \mathcal{C}_{/y}$ induced by the codomain functor $\mathcal{C}^{\Delta^1} \to \mathcal{C}$ is a categorical equivalence and admits a section for all $f : x \to y$.

Definition 4.9. A finitely complete ∞-category $\mathcal{C}$ is *locally cartesian closed (lcc)* if for all morphisms $f : x \to y$, some (hence every) choice of pullback functor $f^* : \mathcal{C}_{/y} \to (\mathcal{C}^{\Delta^1})^{\mathrm{pb}}_{/f} \to \mathcal{C}_{/x}$ has a right adjoint.

Proposition 4.10. *Let $\mathcal{C}$ be finitely complete ∞-category. Then $\mathcal{C}$ is lcc if and only if for all composable morphisms g and f in $\mathcal{C}$, there exists a dependent product of f and g.*

Proof. Let $f : x \to y$ be a map in $\mathcal{C}$ and let $f^* : \mathcal{C}_{/y} \to (\mathcal{C}^{\Delta^1})^{\mathrm{pb}}_{/f} \to \mathcal{C}_{/x}$ be a choice of pullback functor. Then f^* has a left adjoint if and only if for each object $g : z \to x$ of $\mathcal{C}_{/x}$, the ∞-category $\mathcal{C}_{f^*,g}$ defined by the pullback square

$$\begin{tikzcd}
\mathcal{C}_{f^*,g} \arrow[r] \arrow[d] & \mathcal{C}_{/y} \arrow[d, "f^*"] \\
(\mathcal{C}_{/x})_{/g} \arrow[r] & \mathcal{C}_{/x}
\end{tikzcd} \tag{4.5}$$

has a terminal object [52, 17.4]. Since all involved marked simplicial sets are ∞-categories, i.e. fibrant, and the map $(\mathcal{C}_{/x})_{/g} \cong \mathcal{C}_{/g} \to \mathcal{C}_{/x}$ is a fibration Lurie [62, Corollary 2.1.2.2], it follows that (4.4) and (4.5) are homotopy pullback squares. The categorical equivalence $\mathcal{C}_{/y} \to (\mathcal{C}^{\Delta^1})^{\mathrm{pb}}_{/f}$ thus induces a categorical equivalence $\mathcal{C}_{f^*,g} \to \mathcal{C}_{f,g}$, which implies that one of the two categories has a terminal object if and only if the other one has a terminal object. $\square$

Definition 4.11. The discrete category I_{lex} of *lex shapes* is given by triples $\mathbf{C} = (C, K, \phi)$, where C and K are finite simplicial sets and $\phi : \gamma(C) \cong \gamma(\Delta^0 \star K)$ is an isomorphism of C with the cone $\Delta^0 \star K$ in $\mathrm{Ho}(\mathrm{sSet}^+)$. There is an evident functor $I_{\text{lex}} \to \mathrm{sSet}^+$ which maps $\mathbf{C} = (C, K, \phi)$ to $C^\flat$; we often suppress application of this functor and consider $\mathbf{C}$ implicitly as object of sSet^+. A *lex-marked simplicial set* is an I_{lex}-marked object of sSet^+.

The discrete category of $I_{\text{lcc}} \supseteq I_{\text{lex}}$ of *lcc shapes* is given by extending I_{lex} as follows. Denote by Pi the simplicial set which can be depicted as follows:

$$\begin{tikzcd}
\cdot \arrow[dr, "g"'] & \cdot \arrow[l, "e"'] \arrow[r, "p_2"] \arrow[d, "p_1"] \arrow[dr] & \cdot \arrow[d, "f_2"] \\
& \cdot \arrow[r, "f_1"'] & \cdot
\end{tikzcd} \tag{4.6}$$

In addition to the objects of I_{lex}, the objects of I_{lcc} are given by tuples (P, ϕ) with P a finite simplicial set and $\gamma(\phi) : \gamma(P) \to \gamma(\mathrm{Pi})$ an isomorphism of P with Pi in $\mathrm{Ho}(\mathrm{sSet}^+)$. We extend the functor $I_{\text{lex}} \to \mathrm{sSet}^+$ to I_{lcc} by mapping (P, ϕ) to $P^\flat$. As before, we suppress application of the functor $I_{\text{lcc}} \to \mathrm{sSet}^+$. An *lcc-marked simplicial set* is an I_{lcc}-marked object of sSet^+.

The category of lcc-marked simplicial sets can be described equivalently as a category of marked objects of lex-marked simplicial sets. Thus we have a sequence of forgetful functors

$$\begin{tikzcd}
(\mathrm{sSet}^+)^{I_{\text{lcc}}} \arrow[r] & (\mathrm{sSet}^+)^{I_{\text{lex}}} \arrow[l, shift right=2, "\flat"'] \arrow[l, shift left=2, "\sharp"] \arrow[r] & \mathrm{sSet}^+ \arrow[l, shift right=2, "\flat"'] \arrow[l, shift left=2, "\sharp"] \arrow[r] & \mathrm{sSet} \arrow[l, shift right=2, "\flat"'] \arrow[l, shift left=2, "\sharp"]
\end{tikzcd} \tag{4.7}$$

with left adjoints $\flat$ given by minimal marking and right adjoints $\sharp$ given by maximal marking. All adjunctions are Quillen. As in the case of sSet and sSet^+, we generally suppress application of the minimal marking functors, so that e.g. Δ^n can denote the minimally lcc-marked n-simplex. The symbols U, $\flat$ and $\sharp$ are used polymorphically and can denote any (composition) of the forgetful, minimal marking and maximal marking functors of (4.7).

I_{lex} and I_{lcc} can be understood as "signatures" of lex and lcc categories; they encode the shape of universal objects we expect to exist in lex and lcc categories, but not their universal property. The universal properties of these objects, the "axioms", are instead encoded as a sets of cofibrations at which we then localize $(\mathrm{sSet}^+)^{I_{\text{lex}}}$ and $(\mathrm{sSet}^+)^{I_{\text{lcc}}}$. This has the effect of reducing the fibrant objects to those with the right lifting property against these cofibrations.

Definition 4.12. Let K be a finite simplicial set. For $n \geq 0$, we denote by $j^n_{\lim K} : A^n_{\lim K} \to B^n_{\lim K}$ the map of lex-marked simplicial sets given as follows. The map of simplicial sets underlying $j^n_{\lim K}$ is the map $\partial \Delta^n \star K \to \Delta^n \star K$. For $n \geq 1$ the maps from $(\Delta^0 \star K, K, \mathrm{id})$ to the subsets $\Delta^{\{n\}} \star K \cong \Delta^0 \star K$ are marked in both $A^n_{\lim K}$ and $B^n_{\lim K}$; in the codomain $B^n_{\lim K}$ this is also the case for $n = 0$.

Now let $\mathbf{C}_i = (C_i, K, \phi_i), i \in \{1,2\}$ be two objects of I_{lex} which agree on the second component K and let $f : C_1 \to C_2$ be a categorical equivalence such that

$$\begin{tikzcd} & \gamma(C_2) \arrow[dr, "\phi_2"] & \\ \gamma(C_1) \arrow[ur] \arrow[rr, "\phi_1"] & & \gamma(\Delta^0 \star K) \end{tikzcd}$$

commutes in the homotopy category of sSet with the Joyal model structure. Denote by $\overrightarrow{f} = \overrightarrow{f}_{\mathbf{C}_1,\mathbf{C}_2}$ the map $(C_2, \{f\}) \to (C_2, \{\mathrm{id}_{C_2}, f\})$ and by $\overleftarrow{f} = \overleftarrow{f}_{\mathbf{C}_1,\mathbf{C}_2}$ the map $(C_2, \{\mathrm{id}_{C_2}\}) \to (C_2, \{\mathrm{id}_{C_2}, f\})$.

The model category Lex of *sketches for finitely complete ∞-categories* is the left Bousfield localization of $(\mathrm{sSet}^+)^{I_{\text{lex}}}$ at the family of morphisms of the form $j^n_{\lim K}$ and morphisms of the form $\overleftarrow{f}$ and $\overrightarrow{f}$.

Definition 4.13. Let $j^{\Lambda^2_2}_{\Pi} : A^{\Lambda^2_2}_{\Pi} \to B^{\Lambda^2_2}_{\Pi}$ be the map of lcc-marked simplicial sets such that $U(A^{\Lambda^2_2}_{\Pi}) = U(B^{\Lambda^2_2}_{\Pi}) = \mathrm{Pi}$ and $U(j^{\Lambda^2_2}_{\Pi})$ is the identity on Pi, with $\mathrm{id} : (\mathrm{Pi}, \mathrm{id}) \to \mathrm{Pi}$ marked in both $A^{\Lambda^2_2}_{\Pi}$ and $B^{\Lambda^2_2}_{\Pi}$, while in $B^{\Lambda^2_2}_{\Pi}$ additionally $(\Delta^0 \star \Lambda^2_2, \Lambda^2_2, \mathrm{id}) \xrightarrow{\cong} \Delta^1 \times \Delta^1 \xrightarrow{s} \mathrm{Pi}$ is marked. Here s denotes the inclusion of the square with boundary given by the edges p_i, f_i of diagram (4.6) into Pi.

Now let $n \geq 0$. We denote by j^n_{Π} the map of lcc-marked simplicial sets given as follows. The underlying simplicial set of A^n_{Π} is the pushout of

$$\partial\Delta^n \star \Delta^1 \hookleftarrow \partial\Delta^n \star \Delta^0 \hookrightarrow (\partial\Delta^n \star \Delta^0) \times \Delta^1$$

where the left map is the inclusion of $\partial\Delta^n \star \Delta^{\{1\}}$ and the right map is the inclusion of $(\partial\Delta^n \star \Delta^0) \times \Delta^{\{0\}}$.

The underlying simplicial set of B^n_{Π} is defined as pushout of the following span:

$$\Delta^n \star \Delta^1 \hookleftarrow \Delta^n \star \Delta^0 \hookrightarrow (\Delta^n \star \Delta^0) \times \Delta^1$$

Note that the difference to the diagram defining $U(A^n_{\Pi})$ is that $\partial\Delta^n$ is replaced by Δ^n. The map j^n_{Π} is induced by the boundary inclusion $\partial\Delta^n \subseteq \Delta^n$ and functoriality of all involved operators.

A^0_{Π} is minimally marked. The underlying simplicial set of B^0_{Π} is canonically isomorphic to the simplicial set Pi as in diagram (4.6), where the left summand $\Delta^0 \star \Delta^1$ corresponds to the triangle given by e, g and p_1, while the right summand $(\Delta^0 \star \Delta^0) \times \Delta^1$ corresponds to the square given by the f_i and p_i. The markings of B^0_{Π} are given by $(\mathrm{Pi}, \mathrm{id}) \to \mathrm{Pi} \cong U(B^0_{\Pi})$ and the inclusion of $(\Delta^0 \star \Lambda^2_2, \Lambda^2_2, \mathrm{id})$ into the right summand $\Delta^0 \star \Lambda^2_2 \cong \Delta^1 \times \Delta^1 \cong (\Delta^0 \star \Delta^0) \times \Delta^1$.

For $n > 0$, the markings in the domain A^n_{Π} are given by the squares $(\Delta^0 \star \Lambda^2_2, \Lambda^2_2, \mathrm{id}) \to (\Delta^{\{i\}} \star \Delta^0) \times \Delta^1 \subseteq (\partial\Delta^n \star \Delta^0) \times \Delta^1$ for $i \in \{0, \ldots, n\}$, and the map $(\mathrm{Pi}, \mathrm{id}) \xrightarrow{\cong} U(B^0_{\Pi}) \to U(A^n_{\Pi})$ induced by the last vertex $\Delta^{\{n\}} \subseteq \Delta^n$.

The markings of B^n_Π for $n > 0$ are defined similarly, i.e. minimally such that j^n_Π preserves markings.

Let $\mathbf{P_1} = (P_1, \phi_1)$ and $\mathbf{P_2} = (P_2, \phi_2)$ be two objects of $I_{\mathrm{lcc}} \setminus I_{\mathrm{lex}}$, and let $f : P_1 \to P_2$ be a categorical equivalence such that

$$\begin{tikzcd} & \gamma(P_2) \arrow[dr, "\phi_2"] & \\ \gamma(P_1) \arrow[ur, "\gamma(f)"] \arrow[rr, "\phi_1"] & & \gamma(\mathrm{Pi}) \end{tikzcd}$$

commutes in the homotopy category. Denote by $\overleftarrow{f} = \overleftarrow{f}_{\mathbf{P}_1,\mathbf{P}_2}$ the map $(P_2, \{f_1\}) \to (P_2, \{f_1, \mathrm{id}\})$ and by $\overrightarrow{\mathbf{f}} = \overrightarrow{\mathbf{f}}_{\mathbf{P}_1,\mathbf{P}_2}$ the map $(P_2, \{\mathrm{id}\}) \to (P_2, \{f_1, \mathrm{id}\})$.

The model category Lcc of *sketches for locally cartesian closed ∞-categories* is the left Bousfield localization of $(\mathrm{sSet}^+)^{I_{\mathrm{lcc}}}$ at the family of morphisms given by the minimally marked morphisms of Definition 4.12, the morphism $j_\Pi^{\Lambda^2_2}$ and morphisms of the form j^n_Π, $\overrightarrow{\mathbf{f}}$ and $\overleftarrow{\mathbf{f}}$.

It follows from the universal property of the left Bousfield localization that Lcc can equivalently be described as a left Bousfield localization of marked objects in Lex, and that the minimal marking functor $(\mathrm{sSet}^+)^{I_{\mathrm{lex}}} \xrightarrow{\flat} (\mathrm{sSet}^+)^{I_{\mathrm{lcc}}}$ extends to a left Quillen functor Lex $\xrightarrow{\flat}$ Lcc. Note that the maximal marking functors $\sharp \mathrm{sSet}^+ \to$ Lex and $\sharp$: Lex $\to$ Lcc are not right Quillen functors because they do not preserve fibrant objects.

Left Bousfield localizations of simplicial model categories are again simplicial. Thus the mapping spaces $\mathrm{Lex}_{\simeq}(X, Y) = (\mathrm{sSet}^+)^{I_{\mathrm{lex}}}_{\simeq}(X, Y)$ and $\mathrm{Lcc}_{\simeq}(X, Y) = (\mathrm{sSet}^+)^{I_{\mathrm{lcc}}}_{\simeq}(X, Y)$ satisfy the pullback power axioms also with respect to the (trivial) cofibrations and fibrations of Lex and Lcc. This is not generally true for other enrichments, such as enrichment over sSet^+, however. A necessary and sufficient condition for sSet^+-enrichment (and, more generally, enrichment over a model category in which all objects are cofibrant) is given by the following lemma:

Lemma 4.14. *Let $\mathcal{M}$ be a model* sSet^+*-category, let W be a set of morphisms in $\mathcal{M}$ and suppose that the left Bousfield localization $W^{-1}\mathcal{M}$ exists. Then $W^{-1}\mathcal{M}$ is* sSet^+*-enriched as model category if and only if the W-local objects of $\mathcal{M}$ are closed under powers by all marked simplicial sets.*

Proof. Powering by cofibrant objects is a right Quillen functor and hence preserves fibrant objects. Since every marked simplicial set is cofibrant, it follows that the W-local fibrant objects of $\mathcal{M}$, i.e. the fibrant objects of $W^{-1}\mathcal{M}$, are stable under powers if $W^{-1}\mathcal{M}$ is a model sSet^+-category.

Conversely, suppose that the fibrant W-local objects are stable under powers, and let us verify the pushout product axiom. Thus let $i : S \to T$ be a cofibration of marked simplicial sets and let $j : X \to Y$ be a cofibration in $W^{-1}\mathcal{M}$. The pushout product $i \,\square\, j$ is defined by commutativity of the diagram

$$\begin{tikzcd}
S \otimes X \arrow[r] \arrow[d, "S\otimes j"'] & T \otimes X \arrow[d, "f"] \arrow[ddr, bend left, "T\otimes j"] & \\
S \otimes Y \arrow[r] \arrow[drr, bend right] & \cdot \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, "i\square j"] & \\
 & & T \otimes Y.
\end{tikzcd}$$

The cofibrations of $\mathcal{M}$ and $W^{-1}\mathcal{M}$ agree, hence $i \,\square\, j$ is a cofibration if both i and j are cofibrations by the pushout product axiom for $\mathcal{M}$. Similarly, if i is furthermore a trivial cofibration, then $i \,\square\, j$ is a trivial cofibration in $\mathcal{M}$ and hence also a trivial cofibration in $W^{-1}\mathcal{M}$. We are thus left with the case where i is a cofibration and j is a trivial cofibration in $W^{-1}\mathcal{M}$.

If $S \otimes j$ is a trivial cofibration in $W^{-1}\mathcal{M}$, then f is a trivial cofibration in $W^{-1}\mathcal{M}$, and if furthermore $T \otimes j$ is a trivial cofibration, then by 2-out-of-3 also $i \,\square\, j$ is a weak equivalence in $W^{-1}\mathcal{M}$. Thus it suffices to show that $S \otimes j$ is a trivial cofibration in $W^{-1}\mathcal{M}$ for all marked simplicial sets S.

Since all marked simplicial sets are cofibrant $S \otimes j$ is a cofibration in $\mathcal{M}$, hence a cofibration in $S^{-1}\mathcal{M}$. Thus we are left with proving that $S \otimes j$ is a weak equivalence in $W^{-1}\mathcal{M}$. This holds if and only if

$$\mathcal{M}_{\simeq}(S \otimes j, Z) : \mathcal{M}_{\simeq}(S \otimes Y, Z) \to \mathcal{M}_{\simeq}(S \otimes X, Z),$$

is a homotopy equivalence for all W-local fibrant objects Z of $\mathcal{M}$. Equivalently, we can show that the isomorphic map

$$\mathcal{M}_{\simeq}(j, Z^S) : \mathcal{M}_{\simeq}(Y, Z^S) \to \mathcal{M}_{\simeq}(X, Z^S),$$

is a homotopy equivalence. This holds because j is a weak equivalence in $W^{-1}\mathcal{M}$ and Z^S is a W-local fibrant object of $\mathcal{M}$ by assumption. □

Lemma 4.15. 1. *Let $\mathcal{C}$ be a fibrant object of* Lex. *Then $U(\mathcal{C}) \in \mathrm{sSet}^+$ is a finitely complete ∞-category such that for $(C, K, \phi) \in \mathrm{Ob}\, I_{\mathrm{lex}}$, a map $(C, K, \phi) \to U(\mathcal{C})$ is marked if and only if the map $\gamma(\Delta^0 \star K) \xrightarrow{\phi^{-1}} \gamma(C) \to \gamma(U(\mathcal{C}))$ in $\mathrm{Ho}\,\mathrm{sSet}^+$ is in the image of a limit cone $\Delta^0 \star K \to U(\mathcal{C})$.*

2. *Let $\mathcal{C}$ be a fibrant object of* Lcc. *Then $U(\mathcal{C}) \in \mathrm{Ob}\,\mathrm{Lex}$ is a fibrant object, hence satisfies the conditions of 1. A map $k : (P, \phi) \to U(\mathcal{C})$ is marked if and only if $\gamma(k) \circ \phi^{-1} : \gamma(\mathrm{Pi}) \to \gamma(U(\mathcal{C}))$ is in the image of a dependent product. That is, there exists a diagram $k' : \mathrm{Pi} \to U(\mathcal{C})$, which we may depict as*

$$\begin{tikzcd}
\cdot & \cdot \arrow[l] \arrow[r, "p_2"] \arrow[d, "p_1"] \arrow[dr] & \cdot \arrow[d, "f_2"] \\
 & \cdot \arrow[r, "f_1"'] \arrow[ul, leftarrow, "g"] & \cdot
\end{tikzcd}$$

such that k' corresponds to a terminal object of $\mathcal{C}_{f_1,g}$ and $\gamma(k) \circ \phi^{-1} = \gamma(k')$.

Proof. 1. First consider markings $(\Delta^0 \star K, K, \mathrm{id}) \to U(\mathcal{C})$. By assumption, $\mathcal{C}$ has the right lifting property against $j^n_{\lim K}$ for all $n \geq 0$. The lifting property against $j^0_{\lim K}$ implies the existence of a cone $\Delta^0 \star K \to U(\mathcal{C})$ marked as a limit over every diagram $p : K \to U(\mathcal{C})$. The lifting property against $j^n_{\lim K}$ for $n \geq 1$ then asserts that every boundary $\partial\Delta^n \to U(\mathcal{C})_{/p}$ whose last vertex is marked as a limit cone admits a filler. Thus marked cones $\Delta^0 \star K \to U(\mathcal{C})$ exist over every finite diagram $K \to U(\mathcal{C})$, and such cones are limit diagrams. Since limit cones are stable under homotopy and markings in the fibrant lex-marked category $\mathcal{C}$ are stable under homotopy, maps $k : (K, \Delta^0 \star K, \mathrm{id}) \to U(\mathcal{C})$ are marked if and only if they are limit cones.

Now consider a general object $\mathbf{C} = (C, K, \phi)$ of I_{lex}. The functor $\gamma : \mathrm{sSet}^+ \to \mathrm{Ho}(\mathrm{sSet}^+)$ is surjective when restricted to hom-sets from cofibrant to fibrant objects. Thus there exists a cospan

$$C \xrightarrow{i} C' \xleftarrow{f} K \star \Delta^0 \tag{4.8}$$

such that $\phi^{-1} = \gamma(i)^{-1} \circ \gamma(f)$, where i is a trivial cofibration and C' is fibrant, i.e. a fibrant replacement of C. Without loss of generality we may assume that $i : C \to C'$ is given by the small object argument, i.e. that it is a transfinite composition of a chain $C = C_0 \to C_1 \to \cdots \to C_n \to \ldots$ where C_{n+1} is got from C_n as pushout of a map in a set J of generating trivial cofibrations. sSet^+ is generated by finite trivial cofibrations, hence each C_n is a finite simplicial set. Since $K \star \Delta^0$ is finite, f factors via some C_{n_0}. Replacing C' in (4.8) with C_{n_0}, we thus assume that C' is finite (though not fibrant). ϕ^{-1} and $\gamma(i)$ are isomorphisms, hence $\gamma(f)$ is an isomorphism and f is a weak equivalence.

Now let $c_0 : (C, \phi, K) \to U(\mathcal{C})$ be an arbitrary map. By lifting c_0 against the trivial cofibration i and composing with f, we obtain a commuting diagram

$$\begin{tikzcd}
C \arrow[r, "c_0"] \arrow[d, "i"'] & U(\mathcal{C}) \\
C' \arrow[ur, dashed, "c_1"] & \\
K \star \Delta^0 \arrow[u, "f"] \arrow[uur, dashed, "c_2"'] &
\end{tikzcd}$$

such that $\gamma(c_0) \circ \phi^{-1} = \gamma(c_2)$. Note that $(C', K, \gamma(f)^{-1})$ is an object of I_{lex}. By the right lifting property of $\mathcal{C}$ against $\overrightarrow{i}$ and $\overleftarrow{i}$ it follows that c_0 is marked if and only if c_1 is marked. Similarly, by considering $\overrightarrow{f}$ and $\overleftarrow{f}$, it follows that c_1 is marked if and only if c_2 is marked. c_2 is marked if and only if is a limit cone. Since being a limit cone is invariant under homotopy, this is the case if and only if $\gamma(c_2) = \gamma(c_2')$ for some limit cone $c_2' : \Delta^0 \star K \to U(\mathcal{C})$. Thus c_0 is marked if and only if c_2 is marked, which is the case if and only if $\gamma(c_2) = \gamma(c_0) \circ \phi^{-1}$ is in the image of a limit cone.

2. The forgetful functor $U : \mathrm{Lcc} \to \mathrm{Lex}$ is a right Quillen functor, so it preserves fibrant objects. The lifting property against j^0_Π implies the existence of a marked map $k : (\mathrm{Pi}, \mathrm{id}) \to U(\mathcal{C})$ under every pair $z \xrightarrow{g} x \xrightarrow{f} y$ of composable morphisms. The lifting property against $j^{\Lambda^2_2}_\Pi$ then implies that the square $\Delta^1 \times \Delta^1 \xrightarrow{s} \mathrm{Pi} \to U(\mathcal{C})$ is a pullback square, hence k corresponds to a vertex of $\mathcal{C}_{f,g}$. The lifting property against j^n_Π for $n \geq 1$ then implies that marked diagrams $(\mathrm{Pi}, \mathrm{id}) \to U(\mathcal{C})$ under g, f are terminal objects of $\mathcal{C}_{f,g}$. Dependent products are defined in terms of a universal property, hence stable under equivalence. Thus precisely the maps $(\mathrm{Pi}, \mathrm{id}) \to U(\mathcal{C})$ which correspond to dependent products are marked.

We reduce the case of a map $k_0 : (P, \phi) \to U(\mathcal{C})$ in $I_{\mathrm{lcc}} \setminus I_{\mathrm{lex}}$ to the case $(\mathrm{Pi}, \mathrm{id})$ as in the proof of point 1: There exists a cospan

$$P \xrightarrow{j} P' \xleftarrow{f} \mathrm{Pi}$$

of finite simplicial sets such that j is a trivial cofibration, f is a weak equivalence and $\phi = \gamma(f)^{-1} \circ \gamma(j)$. $(P', \gamma(f))$ is an object of $I_{\mathrm{lcc}} \setminus I_{\mathrm{lex}}$, and we obtain a commuting diagram

$$\begin{tikzcd}
P \arrow[r, "k_0"] \arrow[d, "j"'] & U(\mathcal{C}) \\
P' \arrow[ur, dashed, "k_1"] & \\
\mathrm{Pi}. \arrow[u, "f"] \arrow[uur, dashed, "k_2"'] &
\end{tikzcd}$$

The right lifting property against $\overleftarrow{i}$, $\overrightarrow{i}$, $\overleftarrow{f}$ and $\overrightarrow{f}$ now implies that k_0 is marked if and only if k_2 is marked, hence the result follows from our earlier classification of marked maps $(\mathrm{Pi}, \mathrm{id}) \to U(\mathcal{C})$. $\square$

Lemma 4.16. *1. Denote for $n \geq 0$ and finite simplicial set K by $\tilde{j}^n_{\lim K} : \tilde{A}^n_{\lim K} \to \tilde{B}^n_{\lim K}$ the morphism of lex-marked simplicial sets which is defined analogously to $j^n_{\lim K}$ but with the alternative join $\diamond$ instead of $\star$. Thus $U(\tilde{j}^n_{\lim K})$ is the inclusion $\partial\Delta^n \diamond K \subseteq \Delta^n \diamond K$. The markings in domain (for $n \geq 1$) and codomain are of the form $(\Delta^0 \diamond K, K, \gamma(\phi)) \to \Delta^{\{n\}} \diamond K$, where $\phi : \Delta^0 \diamond K \to \Delta^0 \star K$ is the canonical comparison map.*

Then Lex *agrees with the localization of* $(\mathrm{sSet}^+)^{I_{\mathrm{lex}}}$ *at the morphisms of the form $\overleftarrow{f}$, $\overrightarrow{f}$ and $\tilde{j}^n_{\lim K}$.*

2. Denote for $n \geq 0$ by $\tilde{j}^n_\Pi : \tilde{A}^n_\Pi \to \tilde{B}^n_\Pi$ the morphism of lcc-marked simplicial sets which is defined analogously to j^n_Π but with the alternative join $\diamond$ instead of $\star$. Thus $U(\tilde{A}^n_\Pi)$ is the pushout of

$$\partial\Delta^n \diamond \Delta^1 \leftarrow \partial\Delta^n \diamond \Delta^0 \hookrightarrow (\partial\Delta^n \diamond \Delta^0) \times \Delta^1$$

and $U(\tilde{B}^n_\Pi)$ is defined by the analogous pushout diagram with Δ^n in place of $\partial\Delta^n$. The limit markings in domain (for $n \geq 1$; $\tilde{A}^0_\Pi$ is minimally marked) and codomain are given by the maps $(\Delta^0 \star \Lambda^2_2, \Lambda^2_2, \mathrm{id}) \to \Delta^1 \times \Delta^1 \cong (\Delta^{\{i\}} \diamond \Delta^0) \times \Delta^1$ for $0 \leq i \leq n$. Let $\widetilde{\mathrm{Pi}} = U(\tilde{B}^0_\Pi)$ and denote by $\phi : \widetilde{\mathrm{Pi}} \to \mathrm{Pi}$ the map induced by the comparison maps $X \diamond Y \to X \star Y$. Since the product functor $- \times \Delta^1$ preserves weak equivalences and the pushouts defining Pi and $\widetilde{\mathrm{Pi}}$ are Reedy cofibrant, ϕ is a weak equivalence. Dependent product markings in $\tilde{A}^n_\Pi$ and $\tilde{B}^n_\Pi$ are given by the maps $(\widetilde{\mathrm{Pi}}, \phi) \to U(A^n_\Pi)$ and $(\widetilde{\mathrm{Pi}}, \phi) \to U(B^n_\Pi)$ induced by the last vertex $\Delta^{\{n\}} \subseteq \Delta^n$.

Then Lcc *agrees with the localization of* $(\mathrm{sSet}^+)^{I_{\mathrm{lcc}}}$ *at the minimally marked morphisms of item 1 and the morphisms of the form $\overleftarrow{\mathbf{f}}$, $\overrightarrow{\mathbf{f}}$, $j^{\cdot\Lambda^2_2}_\Pi$ and $\tilde{j}^n_\Pi$.*

Proof. We only prove 1, the proof of 2 is similar. Working in the localization of $(\mathrm{sSet}^+)^{I_{\mathrm{lex}}}$ at the morphisms of the form $\overleftarrow{f}$ and $\overrightarrow{f}$, we construct a commuting diagram

$$\begin{tikzcd}
\tilde{A}^n_{\lim K} \arrow[r] \arrow[d, "\tilde{j}^n_{\lim K}"] & \bar{A}^n_{\lim K} \arrow[r] \arrow[d, "\bar{j}^n_{\lim K}"] & \hat{A}^n_{\lim K} \arrow[d, "\hat{j}^n_{\lim K}"] & A^n_{\lim K} \arrow[l] \arrow[d, "j^n_{\lim K}"] \\
\tilde{B}^n_{\lim K} \arrow[r] & \bar{B}^n_{\lim K} \arrow[r] & \hat{B}^n_{\lim K} & B^n_{\lim K} \arrow[l]
\end{tikzcd}$$

for every $n \geq 0$ and finite simplicial K in which all horizontal maps are weak equivalences. It follows that in every localization in which $\tilde{j}^n_{\lim K}$ is a weak equivalence also $j^n_{\lim K}$ is a weak equivalence and vice versa.

We explain the construction of the top row of this diagram; the bottom row is constructed similarly. We define $U(\hat{j}^n_{\lim K})$ as the map $U(j^n_{\lim K})$, but equip $\hat{A}^n_{\lim K}$ and $\hat{B}^n_{\lim K}$ with more markings than $A^n_{\lim K}$ and $B^n_{\lim K}$: In addition to the marked maps $(\Delta^0 \star K, K, \mathrm{id}) \xrightarrow{c} U(A^n_{\lim K})$ also the map $c' : (\Delta^0 \diamond, K, \gamma(\phi)) \to \Delta^0 \star K \xrightarrow{c} U(\hat{A}^n_{\lim K})$ is marked (for $n \geq 1$), where $\phi : \Delta^0 \diamond K \to \Delta^0 \star K$ is the canonical comparison map. The evident maps $A^n_{\lim K} \to \hat{A}^n_{\lim K}$ and $B^n_{\lim K} \to \hat{B}^n_{\lim K}$ are pushouts of $\overleftarrow{\phi}$ and hence weak equivalences.

The comparison map $X \diamond Y \to X \star Y$ for all X, Y induces a map $\tilde{A}^n_{\lim K} \to \hat{A}^n_{\lim K}$, and we define $\bar{A}^n_{\lim K}$ as image of this map. The map $U(\tilde{A}^n_{\lim K}) \to U(\bar{A}^n_{\lim K})$ in sSet^+ is a categorical equivalence and it induces an isomorphism of marked objects in the homotopy category of lex-marked objects, so it is a weak equivalence.

$\bar{A}^n_{\lim K} \to \tilde{A}^n_{\lim K}$ is the identity on underlying simplicial sets, but for $\bar{A}^n_{\lim K}$ only $c'(\Delta^0 \diamond K, K, \gamma(\phi)) \to \Delta^0 \star K \to U(\bar{A}^n_{\lim K})$ is marked, while for $\hat{A}^n_{\lim K}$ also $c : (\Delta^0 \star K, K, \mathrm{id}) \to U(\bar{A}^n_{\lim K})$ is marked. Thus $\bar{A}^n_{\lim K} \to \tilde{A}^n_{\lim K}$ is a pushout of $\overrightarrow{\phi}$, i.e. a weak equivalence. □

Lemma 4.17. *Let $\mathcal{C}$ be an ∞-category with a terminal object. Denote for each $n \geq 0$ by $\Delta^n_t(\mathcal{C})$ the full subset of* $\mathrm{sSet}^+_{\simeq}(\Delta^n, \mathcal{C})$ *spanned by the simplices $x : \Delta^n \to \mathcal{C}$ such that $x(\Delta^{\{n\}})$ is a terminal object, and denote similarly by $\partial\Delta^n_t(\mathcal{C})$ the full subset of* $\mathrm{sSet}^+_{\simeq}(\partial\Delta^n, \mathcal{C})$ *spanned by the simplices $x : \partial\Delta^n \to \mathcal{C}$ such that $x(\Delta^{\{n\}})$ is a terminal object (for $n \geq 1$). Then the canonical map $\Delta^n_t(\mathcal{C}) \to \partial\Delta^n_t(\mathcal{C})$ is a trivial Kan fibration.*

Proof. The boundary inclusion $\partial\Delta^n \subseteq \Delta^n$ is a cofibration and $\mathcal{C}$ is fibrant in sSet^+, hence $\mathrm{sSet}^+_{\simeq}(\Delta^n, \mathcal{C}) \to \mathrm{sSet}^+_{\simeq}(\partial\Delta^n, \mathcal{C})$ is a Kan fibration for all n. For $n > 0$, $\Delta^n_t(\mathcal{C}) \to \partial\Delta^n_t(\mathcal{C})$ is a pullback of this map, hence also a Kan fibration. $\Delta^0_t(\mathcal{C})$ is the core of the subcategory of $\mathcal{C}$ spanned by the terminal objects, which is a contractible Kan complex, and $\partial\Delta^0_t(\mathcal{C}) = \Delta^0$. Thus it remains to show that $\Delta^n_t(\mathcal{C}) \to \partial\Delta^n_t(\mathcal{C})$ is a weak equivalence for $n \geq 1$.

$\partial\Delta^1_t(\mathcal{C})$ can be described as the product of $E(\mathcal{C}) = \mathrm{sSet}^+_{\simeq}(\Delta^{\{0\}}, \mathcal{C})$ with the simplicial subset of $E(\mathcal{C})$ spanned by terminal objects. For each terminal object x of $\mathcal{C}$, there is a map $\bar{x} : E(\mathcal{C}) \to \partial\Delta^1_t(\mathcal{C})$ given by the constant map with value x and the identity on $E(\mathcal{C})$. Now

$$\begin{tikzcd} E(\mathcal{C}^{/x}) \arrow[r] \arrow[d] & \Delta^1_t(\mathcal{C}) \arrow[d] \\ E(\mathcal{C}) \arrow[r, "\bar{x}"] & \partial\Delta^1_t(\mathcal{C}) \end{tikzcd}$$

is a pullback square. x being terminal, the map on the left-hand side is a trivial Kan fibration. Every vertex $\Delta^0 \to \partial\Delta^1_t(\mathcal{C})$ factors via $E(\mathcal{C}) \to \partial\Delta^1_t(\mathcal{C})$ for some x, thus Lemma 4.4 applies and $\Delta^1_t(\mathcal{C}) \to \partial\Delta^1_t(\mathcal{C})$ is a trivial Kan fibration.

Now let $n \geq 2$ and assume that the proposition holds in dimensions $< n$. The square

$$\begin{tikzcd} \partial\Delta^{\{0,2,\ldots,n\}} \arrow[r] \arrow[d] & \Delta^{n-1} \arrow[d] \\ \Lambda^n_1 \arrow[r] & \partial\Delta^n \end{tikzcd}$$

is a pushout square, hence

$$\begin{tikzcd} \partial\Delta^n_t(\mathcal{C}) \arrow[r] \arrow[d] & \Delta^{n-1}_t(\mathcal{C}) \arrow[d] \\ (\Lambda^n_1)_t(\mathcal{C}) \arrow[r] & \partial\Delta^{\{0,2,\ldots,n\}}_t(\mathcal{C}) \end{tikzcd} \tag{4.9}$$

is a pullback square. Here $(\Lambda^n_1)_t(\mathcal{C})$ is defined as the simplicial subset of $\mathrm{sSet}^+_{\simeq}(\Lambda^n_1, \mathcal{C})$ spanned by the maps $x : \Lambda^n_1 \to \mathcal{C}$ with $x(\Delta^{\{n\}})$ terminal. The right vertical map in (4.9) is a trivial Kan fibration by the induction hypothesis, hence the map on the left is a trivial Kan fibration. Because $\mathcal{C}$ is an ∞-category and $\Lambda^n_1 \to \Delta^n$ is a trivial cofibration, the map $\mathrm{sSet}^+_{\simeq}(\Delta^n, \mathcal{C}) \to \mathrm{sSet}^+_{\simeq}(\Lambda^n_1, \mathcal{C})$ is a trivial fibration, and, as a pullback of this map, so is $\Delta^n_t(\mathcal{C}) \to (\Lambda^n_1)_t(\mathcal{C})$. By two-out-of-three and the commuting triangle

$$\begin{tikzcd} & \partial\Delta^n_t(\mathcal{C}) \arrow[dr] & \\ \Delta^n_t(\mathcal{C}) \arrow[ur] \arrow[rr] & & (\Lambda^n_1)_t(\mathcal{C}) \end{tikzcd}$$

it follows that the map $\Delta^n_t(\mathcal{C}) \to \partial\Delta^n_t(\mathcal{C})$ is a weak equivalence. □

Proposition 4.18. *Let $\mathcal{C}$ be a sketch for a finitely complete ∞-category. Then $\mathcal{C}$ is fibrant in* Lex *if and only if $U(\mathcal{C}) \in \mathrm{sSet}^+$ is a finitely complete ∞-category such that for $(C, K, \phi) \in \mathrm{Ob}\, I_{\mathrm{lex}}$, a map $(C, K, \phi) \to U(\mathcal{C})$ is marked if and only if the map $\gamma(\Delta^0 \star K) \xrightarrow{\phi^{-1}} \gamma(C) \to \gamma(U(\mathcal{C}))$ in $\mathrm{Ho}\,\mathrm{sSet}^+$ is in the image of a limit cone $\Delta^0 \star K \to U(\mathcal{C})$.*

Proof. Every fibrant object of Lex satisfies the condition by Lemma 4.15. Conversely, let $\mathcal{C}$ be a lex sketch such that $U(\mathcal{C})$ with precisely the finite limit cones marked. We show that $\mathcal{C}$ for $j : A \to B$ any morphism of the form $\overleftarrow{f}$, $\overrightarrow{f}$ or $\tilde{j}^n_{\lim K}$ as in Lemma 4.16, the map

$$\mathrm{Lex}_{\simeq}(j, \mathcal{C}) : \mathrm{Lex}_{\simeq}(B, \mathcal{C}) \to \mathrm{Lex}_{\simeq}(A, \mathcal{C})$$

is a homotopy equivalence. Let first $j = \overleftarrow{f}$ or $j = \overrightarrow{f}$, where $f : C_1 \to C_2$ is a categorical equivalence of finite simplicial sets and $\mathbf{C}_2 = (C_2, K, \phi)$ is an object of I_{lex} (so that $\mathbf{C_1} = (C_1, K, \phi \circ \gamma(f))$ is an object of I_{lex}, and j mediates between $\mathbf{C}_1$-markings and $\mathbf{C}_2$-markings). Then $U(j) \in \mathrm{sSet}^+$ is an isomorphism, hence so is $\mathrm{sSet}^+_{\simeq}(U(j), U(\mathcal{C}))$. The hom-spaces $\mathrm{Lex}_{\simeq}(-, \mathcal{C}) \subseteq \mathrm{sSet}^+_{\simeq}(-, U(\mathcal{C}))$ are given by the full subspaces spanned by vertices corresponding to marking-preserving maps $\mathrm{Hom}_{\mathrm{Lex}}(-, \mathcal{C})$. Thus it suffices to show that $\mathrm{Hom}_{\mathrm{Lex}}(j, \mathcal{C})$ is an isomorphism (i.e. that $\mathcal{C}$ is orthogonal to j), and then also $\mathrm{Lex}_{\simeq}(j, \mathcal{C})$ will be an isomorphism and in particular a homotopy equivalence.

Consider first $j = \overleftarrow{f}$. Then $\mathcal{C}$ is orthogonal to j if and only if whenever $k : C_2 \to U(\mathcal{C})$ is such that $(C_1, K, \phi \circ \gamma(f)) \xrightarrow{f} C_2 \xrightarrow{k} U(\mathcal{C})$ is marked, then also $k : (C_2, K, \phi) \to U(\mathcal{C})$ is marked. By assumption on $\mathcal{C}$, if kf is marked, then $\gamma(kf) \circ (\phi \circ \gamma(f))^{-1} = \gamma(k') : \gamma(\Delta^0 \star K) \to U(\mathcal{C})$ for some limit cone $k' : \Delta^0 \star K \to U(\mathcal{C})$. But then also $\gamma(k') = \gamma(k) \circ \phi^{-1} = \gamma(kf) \circ (\phi \circ \gamma(f))^{-1}$ is in the image of a limit cone. $\mathcal{C}$ is $\overrightarrow{f}$-orthogonal if and only if in the preceding situation, when $k : (C_2, \phi, K) \to U(\mathcal{C})$ is marked, then also $(C_1, \phi \circ \gamma(f), K) \xrightarrow{f} C_2 \xrightarrow{k} U(\mathcal{C})$ is marked. If k is marked, then $\gamma(k) \circ \phi^{-1} = \gamma(k') : \gamma(\Delta^0 \star K) \to \gamma(U(\mathcal{C}))$ for some limit cone $k : \Delta^0 \star K \to U(\mathcal{C})$. But then $\gamma(kf) \circ (\phi \circ \gamma(f))^{-1} = \gamma(k) \circ \phi^{-1}$, hence $kf : (C_1, \phi \circ \gamma(f), K) \to U(\mathcal{C})$ is marked.

Now let $j = \tilde{j}^n_{\lim K}$ for some $n \geq 0$ and finite simplicial set K. j is a cofibration of lcc-marked objects, so $\mathrm{Lex}(j, \mathcal{C})$ is a Kan fibration. Thus we may apply Lemma 4.4 to show that it is a trivial Kan fibration. Let $k : K \to U(\mathcal{C})$ and consider the following diagram:

$$\begin{array}{ccc}
\mathrm{sSet}^+_{\simeq}(\Delta^n, U(\mathcal{C})^{/k}) & \longrightarrow & \mathrm{sSet}^+_{\simeq}(\Delta^n \diamond K, U(\mathcal{C})) \\
\downarrow & & \downarrow \\
\mathrm{sSet}^+_{\simeq}(\partial\Delta^n, U(\mathcal{C})^{/k}) & \longrightarrow & \mathrm{sSet}^+_{\simeq}(\partial\Delta^n \diamond K, U(\mathcal{C})) \\
\downarrow & & \downarrow \\
\Delta^0 & \xrightarrow{\quad k \quad} & \mathrm{sSet}^+_{\simeq}(K, U(\mathcal{C}))
\end{array}$$

By simplicial enrichment of the alternative join/slice adjunction, the outer and lower rectangles are pullbacks, hence so is the upper rectangle. Taking into account the markings of $\tilde{A}^n_{\lim K}, \tilde{B}^n_{\lim K}$ and $\mathcal{C}$, it follows that

$$\begin{array}{ccc}
\Delta^n_t(U(\mathcal{C})^{/k}) & \longrightarrow & \mathrm{Lex}_{\simeq}(\tilde{B}^n_{\lim K}, \mathcal{C}) \\
\downarrow & & \downarrow \\
\partial\Delta^n_t(U(\mathcal{C})^{/k}) & \longrightarrow & \mathrm{Lex}_{\simeq}(\tilde{A}^n_{\lim K}, \mathcal{C})
\end{array}$$

is a pullback square, and the map on the left-hand side is a trivial Kan fibration by Lemma 4.17. Every vertex $p : \tilde{A}^n_{\lim K} \to \mathcal{C}$ of $\mathrm{Lex}_{\simeq}(\tilde{A}^n_{\lim K}, \mathcal{C})$ factors through $\partial\Delta^n_t(U(\mathcal{C})^{/k})$ if we take for k the restriction of $U(p) : \partial\Delta^n \diamond K \to U(\mathcal{C})$ to $K \subseteq \Delta^n \diamond K$. $\square$

Note that the left Bousfield localization of a simplicial category is again simplicial. This is not true for other enrichments, however. Nevertheless, Lex inherits sSet^+-enrichment from $(\mathrm{sSet}^+)^{I_{\mathrm{Lex}}}$:

Corollary 4.19. Lex *is a model* sSet^+*-category.*

Proof. By Lemma 4.14, it suffices to show that the fibrant lex sketches are stable under powers by marked simplicial sets S. By Proposition 4.18, the fibrant lex sketches are the finitely complete ∞-categories $\mathcal{C}$ with precisely the finite limit diagrams marked. By Lurie [62, Corollary 5.1.2.3], a cone $\Delta^0 \star K \to U(\mathcal{C}) = U(\mathcal{C})^S$ is a limit cone if and only if the composites $\Delta^0 \star K \to U(\mathcal{C})^S \xrightarrow{U(\mathcal{C})^s} U(\mathcal{C})^{\Delta^0} = U(\mathcal{C})$ is a limit cone for all vertices $s : \Delta^0 \to S$. By construction of the powers of marked objects in Proposition 4.6, it follows that precisely the finite limit cones are marked in $\mathcal{C}^S$, which is thus a fibrant lex sketch. $\square$

Lemma 4.20. *Let $\mathcal{C}$ be an ∞-category and let $z \xrightarrow{g} x \xrightarrow{f} y$ be a composable pair of morphisms in $\mathcal{C}$. Denote by $(\mathcal{C}^{\Delta^1})^{/f}_{\mathrm{pb}}$ the full subcategory of the slice*

$(\mathcal{C}^{\Delta^1})^{/f}$ *given by the pullback squares of the form*

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \cdot \arrow[d] \\ x \arrow[r, "f"] & y \end{tikzcd}$$

and define $\mathcal{C}^{f,g}$ *by the following pullback diagram:*

$$\begin{tikzcd} \mathcal{C}^{f,g} \arrow[r] \arrow[d] & (\mathcal{C}^{\Delta^1})^{/f}_{\mathrm{pb}} \arrow[d] \\ \mathcal{C}^{/g} \arrow[r] & \mathcal{C}^{/x}. \end{tikzcd}$$

Then the map $\mathcal{C}_{f,g} \to \mathcal{C}^{f,g}$ *induced by the maps*

$$(\mathcal{C}^{\Delta^1})^{\mathrm{pb}}_{/g} \to (\mathcal{C}^{\Delta^1})^{/g}_{\mathrm{pb}} \qquad \mathcal{C}_{/y} \to \mathcal{C}^{/y} \qquad \mathcal{C}_{/f} \to \mathcal{C}^{/f}$$

is a categorical equivalence. In particular, a vertex of $\mathcal{C}_{f,g}$ *is terminal if and only if its image in* $\mathcal{C}^{f,g}$ *is terminal.*

Proof. By Lurie [62, Proposition 2.1.2.1 and Proposition 4.2.1.6], the fibre products defining $\mathcal{C}_{f,g}$ and $\mathcal{C}^{f,g}$ are homotopy fibre products. It follows that the categorical equivalences on the cospans defining the fibre products induce a weak equivalence on fibre products. □

Proposition 4.21. *Let* $\mathcal{C}$ *be a sketch for an lcc* ∞*-category. Then* $\mathcal{C}$ *is fibrant in* Lcc *if and only if* $U(\mathcal{C}) \in \mathrm{Ob}\,\mathrm{sSet}^+$ *is an lcc* ∞*-category, maps* $c : (C, K, \phi) \to U(\mathcal{C})$ *are marked if and only if* $\gamma(c) \circ \phi^{-1}$ *is in the image of a limit cone* $\Delta^0 \star K \to U(\mathcal{C})$*, and maps* $k : (P, \phi) \to U(\mathcal{C})$ *are marked if and only if* $\gamma(k) \circ \phi^{-1}$ *is in the image of a dependent product* $\mathrm{Pi} \to U(\mathcal{C})$*.*

Proof. Every fibrant object of Lcc satisfies the condition by Lemma 4.15. Conversely, let $\mathcal{C}$ be an lcc category with marked finite limit cones and dependent products. Then by by Proposition 4.18, $U(\mathcal{C}) \in \mathrm{Ob}\,\mathrm{Lex}$ is $j^n_{\lim K}$-local for all n, $\overleftarrow{f}_{\mathbf{C_1},\mathbf{C_2}}$-local and $\overrightarrow{f}_{\mathbf{C_1},\mathbf{C_2}}$ for all suitable f, hence, equivalently, $\mathcal{C}$ is $(j^n_{\lim K})^\flat$-local, $(\overleftarrow{f}_{\mathbf{C_1},\mathbf{C_2}})^\flat$-local and $(\overrightarrow{f}_{\mathbf{C_1},\mathbf{C_2}})^\flat$-local.

Thus it remains to show that $\mathcal{C}$ is $\overleftarrow{f}_{\mathbf{P_1},\mathbf{P_2}}$-local, $\overrightarrow{f}_{\mathbf{P_1},\mathbf{P_2}}$-local, $j_\Pi^{\Lambda^2_2}$-local and j^n_Π-local for all $n \geq 0$. $\overleftarrow{f}$-locality and $\overrightarrow{f}$-locality can be proved as in Proposition 4.18. A similar argument as in the proof of Proposition 4.18 shows that $j_\Pi^{\Lambda^2_2}$-locality can be reduced to $j_\Pi^{\Lambda^2_2}$-orthogonality, which is clear since the square $\Delta^0 \star \Lambda^2_2 \xrightarrow{s} \mathrm{Pi} \xrightarrow{k} U(\mathcal{C})$ of a dependent product k is a pullback square.

Thus it remains to show that $\mathcal{C}$ is j^n_Π-local for all $n \geq 0$, or, equivalently by Lemma 4.16, $\tilde{j}^n_\Pi$-local. $\tilde{j}^n_\Pi$ is a cofibration, hence

$$\mathrm{Lcc}(\tilde{j}^n_\Pi, \mathcal{C}) : \mathrm{Lcc}(\tilde{B}^n_\Pi, \mathcal{C}) \to \mathrm{Lcc}(\tilde{A}^n_\Pi, \mathcal{C})$$

is a Kan fibration. We will show that it is a trivial Kan fibration using Lemma 4.4.

Recall that $U(\tilde{B}^n_\Pi) = \Delta^n \diamond \Delta^1 \amalg_{\Delta^n \diamond \Delta^0} (\Delta^n \diamond \Delta^0) \times \Delta^1 \in \mathrm{sSet}^+$, hence

$$\begin{tikzcd}
\mathrm{sSet}^+_\simeq(U(\tilde{B}^n_\Pi), U(\mathcal{C})) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{sSet}^+_\simeq(\Delta^n \diamond \Delta^0, U(\mathcal{C})^{\Delta^1}) \arrow[d] \\
\mathrm{sSet}^+_\simeq(\Delta^n \diamond \Delta^1, U(\mathcal{C})) \arrow[r] & \mathrm{sSet}^+_\simeq(\Delta^n \diamond \Delta^0, U(\mathcal{C}))
\end{tikzcd}$$

is a pullback square. There is an analogous pullback square for $\tilde{A}^n_\Pi$ in place of $\tilde{B}^n_\Pi$, in which the n-simplex Δ^n is replaced by the boundary $\partial\Delta^n$.

Let $z \xrightarrow{g} x \xrightarrow{f} y$ be a composable pair of morphisms in $\mathcal{C}$, corresponding to a map $\langle g, f\rangle : \tilde{A}^0_\Pi \to U(\mathcal{C})$. There are maps $\tilde{A}^0_\Pi \to \tilde{A}^n_\Pi$ and $\tilde{A}^0_\Pi \to \tilde{A}^n_\Pi$ induced by the unique map $\partial\Delta^0 = \emptyset \subseteq \partial\Delta^n \subseteq \Delta^n$ on the components of the pushouts defining $\tilde{A}^0_\Pi, \tilde{A}^n_\Pi$ and $\tilde{B}^n_\Pi$:

$$\begin{tikzcd}
 & \mathcal{C} & \\
\Delta^1 \arrow[ur, "g"] \arrow[d] & \Delta^0 \arrow[l] \arrow[r] \arrow[u, "x"] \arrow[d] & \Delta^0 \times \Delta^1 \arrow[ul, "f"'] \arrow[d] \\
\partial\Delta^n \diamond \Delta^1 \arrow[d] & \partial\Delta^n \diamond \Delta^0 \arrow[l] \arrow[r] \arrow[d] & (\partial\Delta^n \diamond \Delta^0) \times \Delta^1 \arrow[d] \\
\Delta^n \diamond \Delta^1 & \Delta^n \diamond \Delta^0 \arrow[l] \arrow[r] & (\Delta^n \diamond \Delta^0) \times \Delta^1
\end{tikzcd}$$

Taking into account the markings of $\tilde{A}^n_\Pi$, $\tilde{B}^n_\Pi$ and $\mathcal{C}$, it follows that the lower square and the outer rectangle of

$$\begin{tikzcd}
\Delta^n_t(U(\mathcal{C})^{f,g}) \arrow[r] \arrow[d, "k"] & \mathrm{Lcc}_\simeq(\tilde{B}^n_\Pi, \mathcal{C}) \arrow[d] \\
\partial\Delta^n_t(U(\mathcal{C})^{f,g}) \arrow[r, "\overline{\langle g,f\rangle}"] \arrow[d] & \mathrm{Lcc}_\simeq(\tilde{A}^n_\Pi, \mathcal{C}) \arrow[d] \\
\Delta^0 \arrow[r, "\langle g,f\rangle"] & \mathrm{Lcc}_\simeq(\tilde{A}^0_\Pi, \mathcal{C})
\end{tikzcd}$$

are pullbacks, hence so is the upper square. Because $U(\mathcal{C})$ is lcc, k is a trivial fibration. Thus the family of maps $\overline{\langle g, f\rangle}$ for all f, g satisfies the conditions of Lemma 4.4. $\square$

Remark 4.22. In contrast to Lex, the model category Lcc of lcc sketches is *not* enriched over sSet^+ in the model categorical sense: Lcc categories are not closed under powers by arbitrary marked simplicial sets.

Cocone and slice sketches for lex and lcc ∞-categories

Here we show that lex and lcc categories are stable under slicing. The model categorical phrasing of this fact is that the right Quillen functor $\mathrm{sSet}^+_{\Delta^0/} \to \mathrm{sSet}^+$ given by $(x : \Delta^0 \to K) \mapsto K_{/x}$ extends to right Quillen functors $\mathrm{Lex}_{\Delta^0/} \to \mathrm{Lex}$ and $\mathrm{Lcc}_{\Delta^0/} \to \mathrm{Lcc}$.

Note that the join functors $\star, \diamond : \mathrm{sSet}^+ \times \mathrm{sSet}^+ \to \mathrm{sSet}^+$ preserve categorical equivalences in both arguments and hence descend along $\gamma : \mathrm{sSet}^+ \to \mathrm{Ho}(\mathrm{sSet}^+)$ to functors

$$\star_h, \diamond_h : \mathrm{Ho}(\mathrm{sSet}^+) \times \mathrm{Ho}(\mathrm{sSet}^+) \xrightarrow{\cong} \mathrm{Ho}(\mathrm{sSet}^+ \times \mathrm{sSet}^+) \to \mathrm{Ho}(\mathrm{sSet}^+).$$

The natural comparison maps $K \diamond L \to K \star L$ for all marked simplicial sets K and L are weak equivalences, hence induce a natural equivalence $\diamond_h \cong \star_h$.

Definition 4.23. Let A be a lex sketch. The *lex cocone sketch* $A^{\triangleright} \in \mathrm{Ob}\,\mathrm{Lex}$ is given by the underlying marked simplicial set $U(A^{\triangleright}) = U(A) \star \Delta^0$ and the following markings: For every marked map $c : (C, K, \phi) \to U(A)$, the map $c \star \Delta^0 : (C \star \Delta^0, K \star \Delta^0, \phi') \to U(A^{\triangleright})$ is marked in $U(A^{\triangleright})$. Here ϕ' is the composite

$$\gamma(C\star\Delta^0) = \gamma(C)\star\gamma(\Delta^0) \xrightarrow{\phi\star_h\gamma(\Delta^0)} \gamma(\Delta^0\star K)\star_h\gamma(\Delta^0) = \gamma((\Delta^0\star K)\star\Delta^0) \xrightarrow{\cong} \gamma(\Delta^0\star(K\star\Delta^0)).$$

The *alternative lex cocone sketch* $A_{\triangleright} \in \mathrm{Ob}\,\mathrm{Lex}$ is given by the underlying simplicial set $U(A_{\triangleright}) = U(A) \diamond \Delta^0$ and the following markings: For every marked map $c : (C, K, \phi) \to U(A)$, the map $c \diamond \Delta^0 : (C \diamond \Delta^0, K \star \Delta^0, \phi') \to U(A_{\triangleright})$ is marked in $U(A_{\triangleright})$. Here ϕ' is the map

$$\gamma(C\diamond\Delta^0) = \gamma(C)\diamond_h\gamma(\Delta^0) \xrightarrow{\phi\diamond_h\gamma(\Delta^0)} \gamma(\Delta^0\star K)\diamond_h\gamma(\Delta^0) \xrightarrow{\cong} \gamma(\Delta^0\star(K\star\Delta^0))$$

induced by ϕ, the natural isomorphism $\diamond_h \cong \star_h$ and associativity of $\star$.

Now let $x : \Delta^0 \to A$ be an object A. The *lex slice sketch* $A_{/x}$ is given by the underlying marked simplicial set $U(A_{/x}) = U(A)_{/x}$ and the following markings: A map $c : (C, K, \phi) \to U(A_{/x})$ is marked if and only if its transpose $c' : (C \star \Delta^0, K \star \Delta^0, \phi') \to U(A)$ is marked in A. The *alternative lex slice sketch* $A^{/x}$ is given by the underlying marked simplicial set $U(A^{/x}) = U(A)^{/x}$ and the following markings: A map $c : (C, K, \phi) \to U(A_{/x})$ is marked if and only if its transpose $c' : (C \diamond \Delta^0, K \star \Delta^0, \phi') \to U(A)$ is marked in A.

Definition 4.24. Let A be an lcc sketch. The *lcc cocone sketch* $A^{\triangleright}$ is given by the underlying lex sketch $U(A^{\triangleright}) = U(A)^{\triangleright} \in \mathrm{Lex}$ with lcc-markings given by the maps

$$(P, \phi) \xrightarrow{k} U(A) \to U(A)^{\triangleright}$$

for marked $k : (P, \phi) \to U(A)$. The *alternative lcc cocone sketch* $A_{\triangleright} \in \mathrm{Ob}\,\mathrm{Lex}$ is given by the underlying lex sketch $U(A_{\triangleright}) = U(A)_{\triangleright}$ with lcc-markings given

by the maps
$$(P,\phi) \xrightarrow{k} U(A) \to U(A)_{\triangleright}$$
for marked $k :: (P,\phi) \to U(A)$.

Now let $x : \Delta^0 \to A$ be an object A. The *lcc slice sketch* $A_{/x}$ is given by the underlying marked simplicial set $U(A_{/x}) = U(A)_{/x}$ with maps $c : (P,\phi) \to U(A)_{/x}$ marked if and only if

$$(P,\phi) \xrightarrow{c} U(A)_{/x} \to U(A)$$

is marked. The *alternative lcc slice sketch* $A^{/x}$ is given by the underlying lex sketch $U(A^{/x}) = U(A)^{/x}$ with maps $c : (P,\phi) \to U(A)_{/x}$ marked if and only if

$$(P,\phi) \xrightarrow{c} U(A)^{/x} \to U(A)$$

is marked.

Proposition 4.25. *The adjunction* $\mathrm{sSet}^+ \rightleftarrows \mathrm{sSet}^+_{\Delta^0/}$ *given by (alternative) cocone and slice extend to adjunctions* $\mathrm{Lex} \rightleftarrows \mathrm{Lex}_{\Delta^0/}$ *and* $\mathrm{Lcc} \rightleftarrows \mathrm{Lcc}_{\Delta^0/}$ *via the (alternative) lex and lcc cocone and slice constructions. In case of the alternative cocone and alternative slice, also the* sSet^+*-adjunction extends to* sSet^+*-adjunctions of lex and lcc cocone and slice.*

Proof. The 1-categorical adjunctions are by definition. As for the enriched adjunction in case of the alternative cocone and slice, note that the marked simplicial sets $\mathrm{Lex}(X,Y)$ and $\mathrm{Lcc}(X,Y)$ are defined as full marked simplicial subsets of $\mathrm{sSet}^+(U(X),U(Y))$ given by the vertices of marking-preserving maps $X \to Y$. Since the 1-categorical adjunction establishes an isomorphism of the vertices of $\mathrm{Lex}_{\Delta^0/}(X_{\triangleright},(Y,y:\Delta^0\to U(Y)))$ and $\mathrm{Lex}(X,Y^{/y})$, it follows that the isomorphism $\mathrm{sSet}^+_{\Delta^0/}(U(X)\diamond\Delta^0,(U(Y),y:\Delta^0\to U(Y))) \cong \mathrm{sSet}^+(U(X),U(Y)^{/Y})$ restricts to an isomorphism $\mathrm{Lex}_{\Delta^0/}(X_{\triangleright},(Y,y:\Delta^0\to U(Y))) \cong \mathrm{Lex}(X,Y^{/y})$, and similarly for Lcc. ☐

Our next goal is to prove that the slice functors preserve fibrant objects, and then that the cocone functors preserve trivial cofibrations.

Lemma 4.26. *Let* $(f_i : X \to Y_i)_{i\in I}$ *be a family of trivial cofibrations in a model category* $\mathcal{M}$*. Then the map* $f : X \to \coprod_X Y_i$ *is a trivial cofibration.*

Proof. f can be obtained as the coproduct of the trivial cofibrations $\mathrm{id}_X \xrightarrow{f_i} f_i$ in the coslice model category under X. ☐

Lemma 4.27. *Let* A *and* B *be marked simplicial sets. Then the canonical map*

$$A\star\Delta^0 \amalg_{\Delta^0} \Delta^0\star B \to A\star\Delta^0\star B$$

is a trivial cofibration of marked simplicial sets.

Proof. The map is a monomorphism, hence a cofibration, and it reflects markings. Thus it suffices to prove that the map is a categorical equivalence in sSet for simplicial sets A, B.

We first reduce to the case $A = \Delta^m, B = \Delta^n$ by skeletal induction. We thus need to show that the set of simplicial sets A, B for which the proposition holds is closed under coproducts, countable sequential colimits of cofibrations, and under pushouts along cofibrations. More generally, we show closure under colimits over Reedy cofibrant diagrams $I \to \mathrm{sSet}$ in each argument, where I is a Reedy category such that $\mathrm{colim} : \mathrm{sSet}^I \to \mathrm{sSet}$ is a left Quillen functor.

Thus let I be such a Reedy category, let $A : I \to \mathrm{sSet}$ be Reedy cofibrant, and suppose that

$$\psi_i : A(i) \star \Delta^0 \amalg_{\Delta^0} \Delta^0 \star B \to A(i) \star \Delta^0 \star B$$

is a categorical equivalence for all $i \in \mathrm{Ob}\, I$. We need to show that

$$\psi : (\mathrm{colim}\, A) \star \Delta^0 \amalg_{\Delta^0} \Delta^0 \star B \to (\mathrm{colim}\, A) \star \Delta^0 \star B$$

is a categorical equivalence. Join functors and pushout functors, regarded as functors to coslice model categories, are left Quillen functors; in particular, they preserve colimits and Reedy cofibrant diagrams. Thus $\psi = \mathrm{colim}_{i\in I}\, \psi_i$, where we regard $(\psi_i)_{i\in I}$ as natural transformation of Reedy cofibrant diagrams

$$I \xrightarrow{A} \mathrm{sSet} \xrightarrow{-\star\Delta^0} \mathrm{sSet}_{\Delta^0/} \longrightarrow \mathrm{sSet}_{\Delta^0\star B/}$$

(the last functor is the pushout functor along $\Delta^0 \to \Delta^0 \star B$) and

$$I \xrightarrow{A} \mathrm{sSet} \xrightarrow{-\star\Delta^0\star B} \mathrm{sSet}_{\Delta^0\star B/}$$

The forgetful functor $\mathrm{sSet}_{\Delta^0\star B/} \to \mathrm{sSet}$ preserves and reflects weak equivalences. Thus every ψ_i is a weak equivalence in $\mathrm{sSet}_{\Delta^0\star B/}$, hence $\psi = \mathrm{colim}_{i\in I}\, \psi_i$ is a weak equivalence in $\mathrm{sSet}_{\Delta^0\star B/}$, hence it is a categorical equivalence in sSet. The argument for B is dual.

We are thus left with the case of $A = \Delta^m$ and $B = \Delta^n$ for some $m, n \geq 0$. For $m = n = 0$, the map in question is the inclusion of the inner horn $\Lambda^2_1 \subseteq \Delta^2$, which is a trivial cofibration. We proceed by induction over the well-ordering $(m_0, n_0) < (m_1, n_1) \iff m_0 < m_1 \vee n_0 < n_1$. Thus let $m, n \geq 0$ such that the proposition holds for $(m, n-1)$ and for $(m-1, n)$.

Define maps $f_i : \Delta^m \star \Delta^0 \amalg_{\Delta^0} \Delta^0 \star \Delta^n \to F_i$ for $0 \leq i \leq m$ by pushout squares

$$\begin{tikzcd}
\Delta^{\{0,\dots,\hat{\imath},\dots,m\}} \star \Delta^0 \amalg_{\Delta^0} \Delta^0 \star \Delta^n \arrow[r] \arrow[d] & \Delta^{\{0,\dots,\hat{\imath},\dots,m\}} \star \Delta^0 \star \Delta^n \arrow[d] \\
\Delta^m \star \Delta^0 \amalg_{\Delta^0} \Delta^0 \star \Delta^n \arrow[r, "f_i"] \arrow[ur, phantom, "\ulcorner", near end] & F_i
\end{tikzcd}$$

and for $m+2 \leq i \leq m+2+n$ and $i_0 := i-m-2$ by pushout squares

$$\begin{tikzcd}
\Delta^m \star \Delta^0 \amalg_{\Delta^0} \Delta^0 \star \Delta^{\{0,\dots,\hat{i}_0,\dots,n\}} \arrow[r] \arrow[d] & \Delta^m \star \Delta^0 \star \Delta^{\{0,\dots,\hat{i}_0,\dots,n\}} \arrow[d] \\
\Delta^m \star \Delta^0 \amalg_{\Delta^0} \Delta^0 \star \Delta^n \arrow[r, "f_i"] \arrow[ur, phantom, "\ulcorner", near end] & F_i.
\end{tikzcd}$$

By the induction hypothesis, the top arrow in both diagrams are trivial cofibrations, hence so are the f_i. Note that under the isomorphism $\Delta^m \star \Delta^0 \star \Delta^n \cong \Delta^{m+2+n}$, the subobject $\Delta^m \star \Delta^0 \amalg_{\Delta^0} \Delta^0 \star \Delta^n$ is contained in Λ^{m+2+n}_{m+1}. On the other hand, F_i contains the ith face of Δ^{m+2+n} for each i. We thus have a commuting diagram

$$\begin{tikzcd}
\Delta^m \star \Delta^0 \amalg_{\Delta^0} \Delta^0 \star \Delta^n \arrow[r, "f"] \arrow[d, "\cong"] & \operatorname{colim} f_i \arrow[r] \arrow[d, "\cong"] & \Delta^m \star \Delta^0 \star \Delta^n \arrow[d, "\cong"] \\
\Delta^{\{0,\dots,m+1\}} \cup \Delta^{\{m+1,\dots,m+2+n\}} \arrow[r] & \Lambda^{m+2+n}_{m+1} \arrow[r] & \Delta^{m+2+n}.
\end{tikzcd}$$

Here $\operatorname{colim} f_i$ denotes the colimit of the F_i under $\Delta^m \star \Delta^0 \amalg_{\Delta^0} \Delta^0 \star \Delta^n$. f is a trivial cofibration by Lemma 4.26, and $\Lambda^{m+2+n}_{m+1} \subseteq \Delta^{m+2+n}$ is an inner horn inclusion. $\square$

Lemma 4.28. *Let S be a marked simplicial set. Then the canonical map $r : (S \star \Delta^m) \times \Delta^n \to (S \times \Delta^n) \star \Delta^m$ admits a section.*

Proof. Define a proposed section s by commutativity of

$$\begin{tikzcd}
S \star \Delta^m \arrow[d, "="'] & (S \times \Delta^n) \star \Delta^m \arrow[l, "p_1 \star \Delta^m"'] \arrow[r, "p_2 \star \Delta^m"] \arrow[d, "s"] & \Delta^n \star \Delta^m \arrow[d, "f"] \\
S \star \Delta^m & (S \star \Delta^m) \times \Delta^n \arrow[l, "p_1"'] \arrow[r, "p_2"] & \Delta^n
\end{tikzcd}$$

Here f denotes the unique retraction to the inclusion $\Delta^n \hookrightarrow \Delta^n \star \Delta^m$, i.e. via the isomorphism $\Delta^n \star \Delta^m \cong \Delta^{n+m+1}$ induced by $i \mapsto \min(i,n)$.

The identity $rs = \mathrm{id}$ follows from the universal property of the join Rezk [70, Lemma 24.14] from the following observations:

1. The diagram

$$\begin{tikzcd}
(S \times \Delta^n) \star \Delta^m \arrow[r, "s"] \arrow[dr, "p_1 \star \Delta^m"] \arrow[dd] & (S \star \Delta^m) \times \Delta^n \arrow[r, "r"] \arrow[d, "p_1"] & (S \times \Delta^n) \star \Delta^m \arrow[dd] \\
 & S \star \Delta^m \arrow[d] & \\
\Delta^0 \star \Delta^0 \arrow[r, "\mathrm{id}"] & \Delta^0 \star \Delta^0 \arrow[r, "\mathrm{id}"] & \Delta^0 \star \Delta^0
\end{tikzcd}$$

   commutes.

2. Define s_0 via the pullback square

$$\begin{tikzcd}
S \times \Delta^n \arrow[r, "s_0"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S \times \Delta^n \arrow[d] \\
(S \times \Delta^n) \star \Delta^m \arrow[r, "s"] & (S \star \Delta^m) \times \Delta^n.
\end{tikzcd}$$

Then

$$\begin{tikzcd}
S \times \Delta^n \arrow[rr] \arrow[d] \arrow[ddr] & & S \times \Delta^n \arrow[d] \arrow[ddl] \\
(S \times \Delta^n) \star \Delta^m \arrow[rr] \arrow[dddr] & & (S \star \Delta^m) \times \Delta^n \arrow[dddl] \\
& S & \\
& & \\
& S \star \Delta^m &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& S \times \Delta^n \arrow[rr] \arrow[d] \arrow[ddl] & & S \times \Delta^n \arrow[d] \arrow[ddl] \\
& (S \times \Delta^n) \star \Delta^m \arrow[rr] \arrow[ddl] & & (S \star \Delta^m) \times \Delta^n \arrow[ddl] \\
\Delta^n \arrow[rr, "="] \arrow[d] & & \Delta^n \arrow[d, "="'] & \\
\Delta^n \star \Delta^m \arrow[rr, "f"] & & \Delta^n &
\end{tikzcd}$$

commute, hence $s_0 = \mathrm{id}$. In particular,

$$\begin{tikzcd}
S \times \Delta^n \arrow[r, "\mathrm{id}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S \times \Delta^n \arrow[r, "\mathrm{id}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S \times \Delta^n \arrow[d] \\
(S \times \Delta^n) \star \Delta^m \arrow[r, "s"] & (S \star \Delta^m) \times \Delta^n \arrow[r, "r"] & S \times \Delta^n
\end{tikzcd}$$

commutes.

3. The diagram

$$\begin{tikzcd}
\Delta^m \arrow[rr] \arrow[d] \arrow[ddr] & & \Delta^m \times \Delta^n \arrow[d] \arrow[ddl] \\
(S \times \Delta^n) \star \Delta^m \arrow[rr] \arrow[ddr] & & (S \star \Delta^m) \times \Delta^n \arrow[ddl] \\
& \Delta^m \arrow[d] & \\
& S \star \Delta^m &
\end{tikzcd}$$

commutes, hence the top arrow in

$$\begin{tikzcd}
\Delta^m \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \Delta^m \times \Delta^n \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \Delta^m \arrow[d] \\
(S \times \Delta^n) \star \Delta^m \arrow[r, "s"] & (S \star \Delta^m) \times \Delta^n \arrow[r, "r"] & (S \times \Delta^n) \star \Delta^m
\end{tikzcd}$$

is the identity.

$\square$

Lemma 4.29. *Let $\mathcal{C}$ be an ∞-category, let $x \in \mathcal{C}_0$ be an object and let g be a vertex of $((\mathcal{C}_{/x})^{\Delta^1})_0$. Denote by g_0 be the image of g under the map $((\mathcal{C}_{/x})^{\Delta^1}) \to \mathcal{C}^{\Delta^1}$. Then the canonical map*

$$((\mathcal{C}_{/x})^{\Delta^1})_{/g} \to (\mathcal{C}^{\Delta^1})_{/g'}$$

is a categorical equivalence.

Proof. The lemma holds if and only if for all marked simplicial sets S, the canonical map

$$\mathrm{sSet}^{+}_{\simeq}(S, ((\mathcal{C}^{/x})^{\Delta^1})^{/g}) \to \mathrm{sSet}^{+}_{\simeq}(S, (\mathcal{C}^{\Delta^1})^{/g'})$$

is a homotopy equivalence. By the enriched adjunctions maps for slice and cocone, this is equivalent to showing that $\mathrm{sSet}^{+}_{\simeq}(j, \mathcal{C})$ is a weak equivalence, where j is defined by the following pushout diagram:

$$\begin{tikzcd}
\Delta^0 \times \Delta^1 \arrow[r] \arrow[d] & (\Delta^0 \times \Delta^1) \diamond \Delta^0 \arrow[d] \arrow[ddr, bend left] & \\
(S \diamond \Delta^0) \times \Delta^1 \arrow[r] \arrow[drr, bend right] & \cdot \arrow[ul, phantom, very near start, "\ulcorner"] \arrow[dr, dashed, "j"] & \\
& & ((S \diamond \Delta^0) \times \Delta^1) \diamond \Delta^0.
\end{tikzcd}$$

Since product functors, join functors and pushouts of Reedy cofibrant spans preserve categorical equivalences, we can equivalently show that j' defined by

$$\begin{tikzcd}
\Delta^0 \times \Delta^1 \arrow[r] \arrow[d] & (\Delta^0 \times \Delta^1) \star \Delta^0 \arrow[d] \arrow[ddr, bend left] & \\
(S \star \Delta^0) \times \Delta^1 \arrow[r] \arrow[drr, bend right] & \cdot \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "j'"] & \\
 & & ((S \star \Delta^0) \times \Delta^1) \star \Delta^0.
\end{tikzcd}$$

is a categorical equivalence. Categorical equivalences are stable under retracts, hence by Lemma 4.28 it suffices to show that j'' defined by

$$\begin{tikzcd}
\Delta^0 \times \Delta^1 \arrow[r] \arrow[d] & (\Delta^0 \star \Delta^0) \times \Delta^1 \arrow[d] \arrow[ddr, bend left] & \\
(S \star \Delta^0) \times \Delta^1 \arrow[r] \arrow[drr, bend right] & \cdot \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "j''"] & \\
 & & (S \star \Delta^0 \star \Delta^0) \times \Delta^1.
\end{tikzcd}$$

is a categorical equivalence. $- \times \Delta^1$ preserves colimits and categorical equivalences, hence we further reduce to showing that j''' in

$$\begin{tikzcd}
\Delta^0 \arrow[r] \arrow[d] & \Delta^0 \star \Delta^0 \arrow[d] \arrow[ddr, bend left] & \\
S \star \Delta^0 \arrow[r] \arrow[drr, bend right] & \cdot \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "j'''"] & \\
 & & S \star \Delta^0 \star \Delta^0
\end{tikzcd}$$

is a categorical equivalence, which is an instance of Lemma 4.28. $\square$

Proposition 4.30. *Let $\mathcal{C}$ be an ∞-category and let $x : \Delta^0 \to \mathcal{C}$ be an object.*

1. *A cone $c : \Delta^0 \star K = \Delta^0 \star K \to \mathcal{C}_{/x}$ is a limit cone over $c_{|K} : K \to \mathcal{C}_{/x}$ if and only if the corresponding map $c' : \Delta^0 \star K \star \Delta^0 \to \mathcal{C}$ is a limit cone over $c'_{|K \star \Delta^0} : K \star \Delta^0 \to \mathcal{C}$.*

2. *A square $c : \Delta^0 \star \Lambda^2_2 \to \mathcal{C}_{/x}$ is a pullback square in the slice over x if and only if the composite*
$$\Delta^0 \star \Lambda^2_2 \xrightarrow{c} \mathcal{C}_{/x} \to \mathcal{C}$$
is a pullback square.

3. *A map $k : \mathrm{Pi} \to \mathcal{C}_{/x}$ is a dependent product in the slice over x if and only if the composite*
$$\mathrm{Pi} \xrightarrow{k} \mathcal{C}_{/x} \to \mathcal{C}$$
is a dependent product in $\mathcal{C}$.

Proof. 1. Follows from the isomorphism $(\mathcal{C}_{/x})_{/k_{|K}} \cong \mathcal{C}_{/k'_{|K\star\Delta^0}}$.

2. By point 1, $c : \Delta^0 \star \Lambda^2_2 \to \mathcal{C}_{/x}$ is a pullback square (i.e. limit cone over $c_{|\Lambda^2_2}$) if and only if its transpose $c' : \Delta^0 \star \Lambda^2_2 \star \Delta^0 \to \mathcal{C}$ is a limit cone over $c'_{|\Lambda^2_2 \star \Delta^0}$. The image of c in $\mathcal{C}$ can be described in terms of the transpose c' as the composite $\Lambda^2_2 \star \Delta^0 \to \Delta^0 \star \Lambda^2_2 \star \Delta^0 \xrightarrow{c'} \mathcal{C}$. The result thus follows by Lurie [62, Proposition 4.1.1.8] because the inclusion $\Lambda^2_2 \subseteq \Lambda^2_2 \star \Delta^0$ is left anodyne and hence final. (We can obtain $\Lambda^2_2 \star \Delta^0$ from Λ^2_2 by gluing $\Lambda^1_1 \subseteq \Delta^1$, then $\Lambda^2_1 \subseteq \Delta^2$, finally $\Lambda^2_0 \subseteq \Delta^2$.)

3. The tranpose $k' : \mathrm{Pi} \star \Delta^0 \to \mathcal{C}$ of k can be depicted as a diagram

$$\begin{tikzcd}
w_1 \arrow[dr, "g_2"] \arrow[ddr, bend right, "w"'] & \cdot \arrow[l] \arrow[r] \arrow[d] \arrow[dr] & \cdot \arrow[d] \\
g & y_1 \arrow[r, "f_2"] \arrow[d, "y"] \arrow[dr, phantom, "f"] & z_1 \arrow[dl, bend left, "z"] \\
 & x &
\end{tikzcd}$$

in $\mathcal{C}$. The top two rows of this diagram, with x omitted, correspond to the composite $k'' : \mathrm{Pi} \to \mathrm{Pi} \star \Delta^0 \xrightarrow{k} \mathcal{C}$, or, equivalently, to the image of k under $\mathcal{C}_{/x} \to \mathcal{C}$. By point 2, the map $\Delta^0 \star \Lambda^2_2 \star \Delta^0 \to \mathrm{Pi} \star \Delta^0 \xrightarrow{k'} U(\mathcal{C})$ is a limit diagram if and only if $\Delta^0 \star \Lambda^2_2 \to \mathrm{Pi} \xrightarrow{k''} U(\mathcal{C})$ is a limit diagram. Thus k corresponds to a vertex of $(\mathcal{C}_{/x})_{f,g}$ if and only if k'' corresponds to a vertex of $\mathcal{C}_{f_2,g_2}$.

It suffices to show that $(\mathcal{C}_{/x})_{f,g} \to \mathcal{C})_{f_2,g_2}$ is a categorical equivalence since categorical equivalences preserve and reflect terminal objects. Recall that $(\mathcal{C}_{/x})_{f,g}$ and $\mathcal{C}_{f_2,g_2}$ are defined by the following pullback squares:

$$\begin{tikzcd}
(\mathcal{C}_{/x})_{f,g} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & ((\mathcal{C}_{/x})^{\Delta^1})^{\mathrm{pb}}_{/f} \arrow[d] \\
(\mathcal{C}_{/x})_{/g} \arrow[r] & (\mathcal{C}_{/x})_{/y}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}_{f_2,g_2} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathcal{C}^{\Delta^1})^{\mathrm{pb}}_{/f_2} \arrow[d] \\
\mathcal{C}_{/g_2} \arrow[r] & \mathcal{C}_{/y_1}
\end{tikzcd}$$

The two bottom horizontal maps are fibrations, hence both squares are homotopy pullback squares, so it suffices to show that the three maps on lower cospans are categorical equivalences.

The maps $(\mathcal{C}_{/x})_{/y} \to \mathcal{C}_{/y_1}$ and $(\mathcal{C}_{/x})_{/f} \to \mathcal{C}_{/f_2}$ are categorical equivalences by Lurie [62, Propositions 4.1.1.3 (4) and 4.1.1.8] because the inclusions $\Delta^{\{1\}} \subseteq \Delta^1$ and $\Delta^{\{0,1\}} \subseteq \Delta^2$ are left anodyne. Thus it remains to show that the map $((\mathcal{C}_{/x})^{\Delta^1})^{\mathrm{pb}}_{/f} \to (\mathcal{C}^{\Delta^1})^{\mathrm{pb}}_{/f_2}$ is a categorical equivalence.

Note that $((\mathcal{C}_{/x})^{\Delta^1})^{\mathrm{pb}}_{/f}$ and $(\mathcal{C}^{\Delta^1})^{\mathrm{pb}}_{/f_2}$ are themselves defined by a pullback squares

$$
\begin{tikzcd}
(\mathcal{C}_{/x})^{\Delta^1})^{\mathrm{pb}}_{/f} \arrow[r] \arrow[d] & ((\mathcal{C}_{/x})^{\Delta^1})_{/f} \arrow[d] \\
N(\mathcal{E}) \arrow[r] & N(h(((\mathcal{C}_{/x})^{\Delta^1})_{/f}))
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathcal{C}^{\Delta^1})^{\mathrm{pb}}_{/f_2} \arrow[r] \arrow[d] & (\mathcal{C}^{\Delta^1})_{/f_2} \arrow[d] \\
N(\mathcal{E}') \arrow[r] & N(h((\mathcal{C}^{\Delta^1})_{/f_2}))
\end{tikzcd}
$$

where $\mathcal{E} \subseteq h(((\mathcal{C}_{/x})^{\Delta^1})_{/f})$ and $\mathcal{E}' \subseteq h((\mathcal{C}^{\Delta^1})_{/g_2})$ are the full subcategories spanned by the pullback squares. Both subcategories are closed under isomorphisms, hence their inclusions are fibrations (of 1-categories), and the nerve functor N is right Quillen, hence preserves fibrations. Thus the two pullback squares of the last diagram are homotopy pullback squares, and we can again reduce to showing that the maps on lower cospans are categorical equivalences. Both the nerve functor N and the homotopy category functor h preserve weak equivalences. By point 2, $\mathcal{E} \to \mathcal{E}'$ is an equivalence of 1-categories, and $((\mathcal{C}_{/x})^{\Delta^1})_{/f} \to (\mathcal{C}^{\Delta^1})_{/f_2}$ is categorical equivalence by Lemma 4.29. □

Lemma 4.31. *1. Let $\mathcal{C}$ be a fibrant lex sketch and let $x : \Delta^0 \to \mathcal{C}$ be an object of $\mathcal{C}$. Then the alternative lex slice sketch $\mathcal{C}^{/x}$ is fibrant.*

2. Let $\mathcal{C}$ be a fibrant lcc sketch and let $x : \Delta^0 \to \mathcal{C}$ be an object of $\mathcal{C}$. Then the alternative lcc slice sketch $\mathcal{C}^{/x}$ is fibrant.

Proof. 1. We verify the conditions of Proposition 4.18. K be a finite simplicial set and let $k_0 : K \to U(\mathcal{C}_{/x})$. Let $c_0 : (C, K, \phi) \to U(\mathcal{C}^{/x})$ for some $(C, K, \phi) \in I_{\mathrm{lex}}$ be a potentially marked map in the alternative slice. Choose a cospan $C \xrightarrow{j} C' \xleftarrow{f} \Delta^0 \star K$ with j a trivial cofibration and f a categorical equivalence such that $\phi = \gamma(f)^{-1} \circ \gamma(j)$. This data induces c_1 and c_2 in the diagram to the left of the following diagrams:

$$
\begin{tikzcd}
C \arrow[d, "j"'] \arrow[dr, "c_0"] & & \\
C' \arrow[r, "c_1"] & U(\mathcal{C}^{/x}) \arrow[d, phantom, "h"] & U(\mathcal{C}_{/x}) \arrow[l, "\psi"'] \\
\Delta^0 \star K \arrow[u, "f"] \arrow[ur, "c_2"] \arrow[urr, bend right, "c_3"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
C \diamond \Delta^0 \arrow[d, "j\diamond\Delta^0"'] \arrow[dr, bend left, "c_0'"] & \\
C' \diamond \Delta^0 \arrow[r, "c_1'"] & U(\mathcal{C}) \\
(\Delta^0 \star K) \diamond \Delta^0 \arrow[u, "f\diamond\Delta^0"] \arrow[ur, "c_2'"] \arrow[d] \arrow[r, phantom, "h'"] & \\
\Delta^0 \star K \star \Delta^0 \arrow[uur, bend right, "c_3'"'] &
\end{tikzcd}
$$

The canonical comparison map $\psi : U(\mathcal{C}_{/x}) \to U(\mathcal{C}^{/x})$ is a categorical equivalence, $\Delta^0 \star K$ is cofibrant and $U(\mathcal{C}_{/x})$ is an ∞-category. Thus $\gamma(\psi)^{-1} \circ \gamma(c_2)$ has a preimage c_3 under γ, and there exists a homotopy $h : \Delta^1 \to \mathrm{sSet}^+_{\simeq}(\Delta^0 \star K, U(\mathcal{C}^{/x}))$ from c_2 to $\psi \circ c_3$. The transposes of the c_i along the cocone/slice adjunctions are depicted on the right of the previous diagram. Note that the adjunction for the alternative cocone and alternative slice is enriched, hence the edge $h : \Delta^1 \to \mathrm{sSet}^+_{\simeq}(\Delta^0 \star K, U(\mathcal{C}^{/x}))$ corresponds to an edge $h' : \Delta^1 \to \mathrm{sSet}^+_{\Delta^0/}((\Delta^0 \star K) \diamond \Delta^0, U(\mathcal{C}))$. We now conclude that $\mathcal{C}^{/x}$ is a fibrant lex sketch with the equivalences

$$
\begin{aligned}
&c_0 \text{ is marked} \iff c_0' \text{ is marked} \iff c_3' \text{ is a limit cone} \\
\iff &c_3 \text{ is a limit cone} \iff c_2 \text{ is a limit cone.}
\end{aligned}
$$

2. Immediate from point 1, Proposition 4.21 and point 3 of Proposition 4.30. $\square$

Proposition 4.32. *1. Let $j : A \to B$ be a (trivial) cofibration of lex sketches. Then $j^{\triangleright} : A^{\triangleright} \to B^{\triangleright}$ and $j_{\triangleright} : A_{\triangleright} \to B_{\triangleright}$ are (trivial) cofibrations of lex sketches.*

2. Let $j : A \to B$ be a (trivial) cofibration of lcc sketches. Then $j^{\triangleright} : A^{\triangleright} \to B^{\triangleright}$ and $j_{\triangleright} : A_{\triangleright} \to B_{\triangleright}$ are (trivial) cofibrations of lcc sketches.

In particular, the (alternative) cocone/slice adjunctions of Proposition 4.25 are (simplicial) Quillen adjunctions $\mathrm{Lex} \rightleftarrows \mathrm{Lex}_{\Delta^0/}$ *and* $\mathrm{Lcc} \rightleftarrows \mathrm{Lcc}_{\Delta^0/}$.

Proof. Let $\mathcal{M}$ be one of Lex or Lcc, and let j be a cofibration, i.e. a monomorphism, in $\mathcal{M}$. Since both $\star$ and $\diamond$ preserve cofibrations, $j^{\triangleright}$ and $j_{\triangleright}$ are cofibrations. Now assume that j is a trivial cofibration. There is a commuting diagram

$$
\begin{tikzcd}
A_{\triangleright} \arrow[r, "\sim"] \arrow[d, "j_{\triangleright}"'] & A^{\triangleright} \arrow[d, "j^{\triangleright}"] \\
B_{\triangleright} \arrow[r, "\sim"] & B^{\triangleright},
\end{tikzcd}
$$

in which the horizontal arrows are weak equivalences in $\mathcal{M}$, hence by 2-out-of-3 it suffices to show that $j_\triangleright$ is a trivial cofibration. Thus we have to show that the Kan fibration

$$\mathcal{M}_\simeq(j_\triangleright, \mathcal{C}) : \mathcal{M}_\simeq(B_\triangleright, \mathcal{C}) \to \mathcal{M}_\simeq(A_\triangleright, \mathcal{C})$$

is in fact a trivial Kan fibration for all fibrant sketches $\mathcal{C}$. For this it suffices to show that the fibre of $\mathcal{M}_\simeq(j_\triangleright, \mathcal{C})$ over every vertex $a : \Delta^0 \to \mathcal{M}_\simeq(A_\triangleright, \mathcal{C})$, which can be identified with a map $a : A_\triangleright \to \mathcal{C}$, is contractible.

Let $x : \Delta^0 \to A_\triangleright \to \mathcal{C}$ be the restriction of a to the cocone point. Then by Proposition 4.25 the lower and outer rectangles in

$$\begin{tikzcd}
\mathcal{M}_\simeq(B, \mathcal{C}^{/x}) \arrow[r] \arrow[d, "\mathcal{M}_\simeq(j, \mathcal{C}^{/x})"'] & \mathcal{M}_\simeq(B_\triangleright, \mathcal{C}) \arrow[d, "\mathcal{M}_\simeq(j_\triangleright, \mathcal{C})"] \\
\mathcal{M}_\simeq(A, \mathcal{C}^{/x}) \arrow[r] \arrow[d] & \mathcal{M}_\simeq(A_\triangleright, \mathcal{C}) \arrow[d] \\
\Delta^0 \arrow[r, "x"] & \mathcal{M}_\simeq(\Delta^0, \mathcal{C})
\end{tikzcd}$$

are pullback squares, hence so is the upper square. $\mathcal{M}_\simeq(j, \mathcal{C}_{/x})$ is a trivial Kan fibration because j was assumed to be a trivial cofibration and $\mathcal{C}_{/x}$ is fibrant by Lemma 4.31. As p factors via $\mathcal{M}_\simeq(A, \mathcal{C}_{/x})$, the pullback of $\mathcal{M}_\simeq(j_\triangleright, \mathcal{C})$ along p is the pullback of a trivial Kan fibraton and hence itself a trivial Kan fibration. $\square$

4.3 Strict ∞-categories

In this section, we consider model categories of *strict* (lex, lcc) ∞-categories. In contrast to the usual ∞-categories, i.e. simplicial sets with certain right lifting *properties*, strict ∞-categories are equipped with canonical lifts witnessing the lifting property. Their morphisms are those maps of simplicial sets which preserve the witnesses up to equality; usually maps of simplicial sets will preserve such witnesses only up to contractible homotopy. Surprisingly, the model categories of strict ∞-categories we consider are Quillen equivalent to the model categories of sketches, and in particular present the same higher category.

The technical tool for defining our model categories of strict ∞-categories is the formalism of algebraic weak factorization systems and algebraic fibrancy, which applies to general accessible model categories. This is the topic of Subsection 4.3. In Subsection 4.3 we then go on to show that the model category of strict lex (or lcc) ∞-categories admits the structure of a model of dependent type theory with weak intensional identity types and weak finite product types. Finally, in Subsection 4.3, we show that the slice functors on sketches can be extended to functors on strict ∞-categories.

Simplicial-algebraically fibrant objects

Here we shall adapt the notion of algebraically fibrant object in a model category to the simplicially enriched setting. Let $\mathcal{C}$ be a combinatorial category and let J be a set of morphisms in $\mathcal{C}$. Recall that J cofibrantly generates an *algebraic weak factorization system* [15] (L, R) via Garner's small object argument. L is a comonad on the arrow category $\mathcal{C}^{\to}$ while R is a monad on $\mathcal{C}^{\to}$. L and R assign to maps $f : X \to Y$ functorial factorizations

$$\begin{tikzcd} & E(f) \arrow[dr, "R(f)"] & \\ X \arrow[ur, "L(f)"] \arrow[rr, "f"] & & Y \end{tikzcd}$$

such that $R(f)$ has the right lifting property with respect to J and $L(f)$ has the left lifting property with respect to all morphisms with the right lifting property against J.

The category $R-\mathrm{Alg}$ of algebras over the monad R can be described equivalently as the category $J^{\pitchfork}$ of morphisms with J-lifting operations. The objects of $J^{\pitchfork}$ are pairs (f, ℓ_f), where $f : X \to Y$ is a map in $\mathcal{C}$ and ℓ_f is an operation assigning to commuting squares

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "J\ni j"'] & X \arrow[d, "f"] \\ B \arrow[r, "b"'] \arrow[ur, dashed, "\ell_f(a,b)"] & Y \end{tikzcd} \tag{4.10}$$

with $j : A \to B$ in J a diagonal lift $\ell_f(a, b)$ as indicated.

R restricts to a monad R_{Ob} on morphisms with terminal codomain, i.e. to a monad on $\mathcal{C}$. By the characterization of R-Alg as $J^{\pitchfork}$, the category of algebras over R_{Ob} can be described as given by objects X with operations ℓ_X assigning solutions to lifting problems

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "J\ni j"'] & X \\ B. \arrow[ur, dashed, "\ell_X(a)"'] & \end{tikzcd} \tag{4.11}$$

If $\mathcal{C} = \mathcal{M}$ is a combinatorial model category and J is a set of generating trivial cofibration, then $R_{\mathrm{Ob}}-\mathrm{Alg} = \mathrm{Alg}(\mathcal{M})$ is known as category of *algebraically fibrant objects* [66]. Where fibrant objects in $\mathcal{M}$ are objects for which lifts as in (4.11) merely *exist* and are preserved under morphisms in $\mathcal{M}$ only up to contractible homotopy, the algebraically fibrant objects are equipped with canonical choices of lifts, and their morphisms preserve lifts up to equality. The forgetful functor $G : \mathrm{Alg}(\mathcal{M}) \to \mathcal{M}$ has a left adjoint F. It can be shown that

1. $\mathrm{Alg}(\mathcal{M})$ carries the structure of a model category such that G preserves and reflects weak equivalences and fibrations,

2. (F, G) is a Quillen equivalence, and

3. the components $\eta_X : X \to G(F(X))$ are trivial cofibrations for all X in $\mathcal{M}$.

Thus $\mathrm{Alg}(\mathcal{M})$ can be seen as a more algebraic presentation of the higher category presented by $\mathcal{M}$ in which all objects are fibrant.

It is shown in Bidlingmaier [10, Lemma 17] that if $\mathcal{M}$ is a groupoidal model category (i.e. enriched as model category over the category of groupoids with its canonical model category structure), then $\mathrm{Alg}(\mathcal{M})$ is groupoidal. One takes as groupoids of maps $(X, \ell_X) \to (Y, \ell_Y)$ the full subgroupoid of maps $X \to Y$ spanned by the morphisms of algebraically fibrant objects. This, however, does not transfer to the simplicially enriched case, where one would take the full simplicial subset spanned by the vertices corresponding to maps of algebraically fibrant objects. With this definition, it is not clear how one defines simplicial powers. This mismatch between the groupoidal and simplicial cases arises because certain lifts exist uniquely in the 2-truncated groupoidal case but not in the simplicial case.

The solution to this problem is to adapt the 1-categorical definitions of the algebraic weak factorization system (L, R) and the J-lifting operations. Indeed, under suitable conditions on a $\mathcal{V}$-enriched category $\mathcal{C}$, Garner's small object argument can be generalized to the enriched setting, resulting in a $\mathcal{V}$-enriched comonad L and $\mathcal{V}$-enriched monad R. In the 1-categorical case, lifting operations as in (4.10) on a morphism $f : X \to Y$ can be described as sections to the canonical maps

$$\mathrm{Hom}(B, X) \to \mathrm{Hom}(A, X) \times_{\mathrm{Hom}(A,Y)} \mathrm{Hom}(B, Y) = \mathrm{Hom}_{\mathcal{C}^{\to}}(j, f)$$

on hom-sets for all $j : A \to B$ in J, hence the natural generalization to the enriched setting is to demand a section to the map

$$\mathcal{C}(B, X) \to \mathcal{C}(A, X) \times_{\mathcal{C}(A,Y)} \mathcal{C}(B, Y) = \mathcal{C}^{\to}(j, f)$$

in $\mathcal{V}$. We denote the resulting category of maps equipped with $\mathcal{V}$-enriched lifting operations by $J^{\pitchfork}$. As in the unenriched setting, the categories $(R\text{–}\mathrm{Alg})_0$ of algebras over R and $(J^{\pitchfork})_0$ are equivalent (in fact, isomorphic) as 1-categories.

Note that R is a monad enriched over $\mathcal{V}$, hence $R\text{–}\mathrm{Alg}$ is naturally $\mathcal{V}$-enriched. Riehl conjectures [71, Remark 13.4.3] that $J^{\pitchfork}$ admits $\mathcal{V}$-enrichment as follows. Consider first for fixed $j \in J$ and $(f, \ell_f), (g, \ell_g) \in J^{\pitchfork}$ the (non-commutative) diagram

$$\begin{tikzcd}
& \mathcal{C}^{\to}(j,g) \arrow[dr] & \\
\mathcal{C}^{\to}(j,f)\times\mathcal{C}^{\to}(f,g) \arrow[ur] \arrow[dr] & & \mathcal{C}(j_1,g_0) \\
& \mathcal{C}(j_1,f_0)\times\mathcal{C}(f_0,g_0) \arrow[ur] &
\end{tikzcd}$$

induced the enriched lifting operations $\ell_f : \mathcal{C}^{\to}(j,f) \to \mathcal{C}(j_1,f_0)$ and $\ell_g : \mathcal{C}^{\to}(j,g) \to \mathcal{C}(j_1,g_0)$. Here $h \mapsto h_0$ denotes the domain functor $\mathcal{C}^{\to} \to \mathcal{C}$ and $h \mapsto h_1$ denotes the codomain functor. Taking transposes along the product/exponential adjunction with $\mathcal{C}^{\to}(f,g)$ and the product over all $j \in J$, we obtain two maps

$$\mathcal{C}^{\to}(f,g) \rightrightarrows \prod_{j\in J} \mathcal{V}(\mathcal{C}^{\to}(j,f),\mathcal{C}(j_1,g_0)). \tag{4.12}$$

Riehl proposes to define the mapping object $J^{\underline{\pitchfork}}(f,g)$ as equalizer of this diagram, and conjectures that the resulting $\mathcal{V}$-enriched category is isomorphic to $R-$Alg, extending the isomorphism of underlying categories. This is indeed the case, and the proof is our first goal in this section.

Proposition 4.33. *The hom-objects defined as equalizers of* (4.12) *are closed under identities and composition in* $\mathcal{C}^{\to}$, *so that* $J^{\underline{\pitchfork}}$ *can be regarded as enriched category.*

Proof. Closure under identities is clear. Let $f,g,h \in \operatorname{Ob} J^{\underline{\pitchfork}}$, and let $j \in J$. Closure under compositions follows after taking transposes along the product/-exponential adjunction from the commutativity of

$$\begin{tikzcd}
& \mathcal{C}^{\to}(j,f)\times\mathcal{C}^{\to}(f,h) \arrow[r] & \mathcal{C}^{\to}(j,h) \arrow[ddr] & \\
& \mathcal{C}^{\to}(j,g)\times J^{\underline{\pitchfork}}(g,h) \arrow[r] \arrow[ur] & \mathcal{C}(j_1,g_0)\times\mathcal{C}(g_0,h_0) \arrow[dr] & \\
\mathcal{C}^{\to}(j,f)\times J^{\underline{\pitchfork}}(f,g)\times J^{\underline{\pitchfork}}(g,h) \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & & \mathcal{C}(j_1,h_0) \\
& \mathcal{C}(j_1,f_0)\times\mathcal{C}(f_0,g_0)\times\mathcal{C}(g_0,h_0) \arrow[r] \arrow[uur] & \mathcal{C}(j_1,f_0)\times\mathcal{C}(f_0,h_0) \arrow[ur] & \\
& \mathcal{C}^{\to}(j,f)\times\mathcal{C}^{\to}(f,h) \arrow[ur] & &
\end{tikzcd}$$

$\square$

Lemma 4.34. *Let $X \in \operatorname{Ob}\mathcal{V}$ and let $(g, \ell_g) \in \operatorname{Ob} J^{\pitchfork}$. Then the power g^X in $\mathcal{C}^{\to}$ admits $J^{\pitchfork}$-structure ℓ_{g^X}, and the natural isomorphisms*

$$\mathcal{V}(X, \mathcal{C}^{\to}(f, g)) \cong \mathcal{C}^{\to}(f, g^X)$$

for all morphisms f in $\mathcal{C}$ restrict to natural isomorphisms

$$\mathcal{V}(X, J^{\pitchfork}((f, \ell), (g, \ell_g))) \cong J^{\pitchfork}((f, \ell), (g^X, \ell_{g^X})).$$

for all $(f, \ell_f) \in \operatorname{Ob} J^{\pitchfork}$.

Proof. We define a lifting structure ℓ_{g^X} on g^X via natural maps

$$\begin{aligned}
&\operatorname{Hom}(Y, \mathcal{C}^{\to}(j, g^X)) \\
&\cong \operatorname{Hom}(Y \times X, \mathcal{C}^{\to}(j, g)) \\
&\to \operatorname{Hom}(Y \times X, \mathcal{C}^{\to}(j_1, g_0)) \\
&\cong \operatorname{Hom}(Y, \mathcal{C}^{\to}(j_1, (g_0)^X)) \\
&\cong \operatorname{Hom}(Y, \mathcal{C}^{\to}(j_1, (g^X)_0))
\end{aligned}$$

for all $Y \in \operatorname{Ob}\mathcal{V}$ and the Yoneda lemma.

We use again the Yoneda lemma to show that the universal property of g^X restricts to a universal property of (g^X, ℓ_{g^X}). Thus let $(f, \ell_X) \in \operatorname{Ob} J^{\pitchfork}$, let $j \in J$ and let $Y \to \mathcal{C}^{\to}(f, g^X)$. Then

$$\begin{tikzcd}
 & \mathcal{C}^{\to}(j, g^X) \arrow[dr, "\ell_{g^X}"] & \\
\mathcal{C}^{\to}(j, f) \times Y \arrow[ur] \arrow[dr] & & \mathcal{C}^{\to}(j_1, (g_0)^X) \\
 & \mathcal{C}(j_1, f_0) \times \mathcal{C}(f_0, (g^X)_0) \arrow[ur] &
\end{tikzcd}$$

commutes if and only if

$$\begin{tikzcd}
 & \mathcal{C}^{\to}(j, g) \arrow[dr, "\ell_g"] & \\
\mathcal{C}^{\to}(j, f) \times Y \times X \arrow[ur] \arrow[dr] & & \mathcal{C}^{\to}(j_1, g_0) \\
 & \mathcal{C}(j_1, f_0) \times \mathcal{C}(f_0, g_0) \arrow[ur] &
\end{tikzcd}$$

commutes if and only if

$$Y \rightrightarrows \mathcal{V}(X, \mathcal{V}(\mathcal{C}^{\to}(j, f), \mathcal{C}^{\to}(j_1, g_0)))$$

commutes. Since $\mathcal{V}(-,-)$ commutes with limits in the second argument, it follows that $Y \to \mathcal{C}^{\to}(f, g^X)$ is valued in $J^{\pitchfork}((f, \ell_f), (g^X, \ell_{g^X}))$ if and only if $Y \to \mathcal{V}(X, \mathcal{C}^{\to}(f, g))$ is valued in $\mathcal{V}(X, J^{\pitchfork}((f, \ell_f), (g, \ell_g)))$. $\square$

Proposition 4.35. *The canonical isomorphism $(R\text{–}\mathrm{Alg})_0 \cong (J^{\pitchfork})_0$ of underlying 1-categories extends to an isomorphism of $R\text{–}\mathrm{Alg} \cong J^{\pitchfork}$ of $\mathcal{V}$-enriched categories.*

Proof. Recall that the map $(R\text{–}\mathrm{Alg})_0 \to (J^{\pitchfork})_0$ of underlying 1-categories is given on R-Algebras $r_f : R(f) \to f$ by endowing them with lifts according to the diagram

$$\begin{tikzcd}
\cdot \arrow[rrr, "a"] \arrow[dr, "L(j)"] \arrow[dd, "j"'] & & & \cdot \arrow[dd, "f"] \\
& E(j) \arrow[r, "E(a,b)"] & E(f) \arrow[ur, "r_f"'] \arrow[dr, "R(f)"] & \\
\cdot \arrow[ur, "c_j"] \arrow[rrr, "b"'] & & & \cdot
\end{tikzcd}$$

for all $j \in J$. Here E denotes the composite of $R : \mathcal{C}^{\to} \to \mathcal{C}^{\to}$ with the domain functor $\mathcal{C}^{\to} \to \mathcal{C}$, or, equivalently, the composite of L with the codomain functor. $c_j : j \to L(j)$ denotes the canonical L-comonad structure on $j \in J$. Since (L, R) is an enriched weak factorization system, enriched lifting functions on f can be constructed similarly as composite

$$\mathcal{C}^{\to}(j, f) \longrightarrow \mathcal{C}(E(j), E(f)) \xrightarrow{r_f \circ - \circ c_j} \mathcal{C}(j_1, f_0),$$

which defines the isomorphism $(R\text{–}\mathrm{Alg})_0 \xrightarrow{\sim} (J^{\pitchfork})_0$.

Similarly to the powers in $J^{\pitchfork}$ due to Lemma 4.34, the powers $(g, m_g)^X = (g^X, m_{g^X})$ of R-algebras (g, m_g) are given by a pointwise construction via the Yoneda lemma. It follows that the functor $(R\text{–}\mathrm{Alg})_0 \to (J^{\pitchfork})_0$ preserves powers. We can thus extend this 1-categorical functor to a $\mathcal{V}$-enriched functor via the Yoneda lemma from the isomorphisms

$$\begin{aligned}
&\mathrm{Hom}(X, R\text{–}\mathrm{Alg}((f, m_f), (g, m_g))) \\
\cong\; &\mathrm{Hom}_{R\text{–}\mathrm{Alg}}((f, m_f), (g^X, m_{g^X})) \\
\cong\; &\mathrm{Hom}_{J^{\pitchfork}}((f, \ell_f), (g^X, \ell_{g^X})) \\
\cong\; &\mathrm{Hom}(X, J^{\pitchfork}((f, \ell_f), (g, \ell_g)))
\end{aligned}$$

where ℓ_f and ℓ_g are the enriched J-lifting structures induced by R-algebra structure m_f and m_g, and X is an arbitrary object of $\mathcal{V}$. $\square$

Definition 4.36. Let $\mathcal{M}$ be a combinatorial model $\mathcal{V}$-category, and let J be set of trivial cofibrations in $\mathcal{M}$. The category $\mathrm{Alg}_J(\mathcal{M})$ of *algebraically (partially) fibrant objects with respect to J* is the full subcategory of $J^{\pitchfork}$ given by the morphisms $X \to 1$ to a terminal object with simplicially enriched lifting operation against all $j \in J$. We denote the resulting adjunction of $\mathcal{V}$-categories by $F : \mathcal{M} \rightleftarrows: \mathrm{Alg}_J(\mathcal{M}) : G$.

If J is not specified, then $\mathrm{Alg}(\mathcal{M}) = \mathrm{Alg}_J(\mathcal{M})$ denotes the category of *algebraically fibrant objects* with respect to a set J of trivial cofibrations fixed once and for all for $\mathcal{M}$ such that

- J is set of representatives under isomorphism of trivial cofibrations $j : A \to B$ with κ-small domain and codomain for some infinite cardinal κ, and
- J is a set of generating trivial cofibration.

The category $\mathrm{Alg}_J(\mathcal{M})$ as defined here for combinatorial *enriched* model categories $\mathcal{M}$ should not be confused with the notion considered in Nikolaus [66] or Bourke [14]: There, $\mathcal{M}$ is not enriched, and consequently $\mathrm{Alg}_J(\mathcal{M})$ is defined in terms of $J^{\pitchfork}$ instead of $J^{\underline{\pitchfork}}$. Furthermore, we allow J to be an arbitrary set of trivial cofibrations, whereas otherwise J is assumed to be a generating set of trivial cofibrations. In our instantiations of Definition 4.36, however, J is always a generating set.

Note that, by Proposition 4.35, the objects of $\mathrm{Alg}_J(\mathcal{M})$ can equivalently be described as objects X of $\mathcal{M}$ equipped with R-algebra structure on the unique map $X \to 1$ to a terminal object.

Lemma 4.37. *Let $\mathcal{M}$ be a combinatorial model category and let $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ be an adjunction with a locally presentable category $\mathcal{N}$. Suppose the following holds:*

1. *For every object X in $\mathcal{N}$, there exists a morphism $\eta_X : X \to R(X)$ such that $G(\eta_X)$ is a weak equivalence and $G(R(X))$ is fibrant.*
2. *For every morphism $f : X \to Y$, there exists a morphism $R(f) : R(X) \to R(Y)$ such that*
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ R(X) & \xrightarrow{R(f)} & R(Y) \end{array}$$
commutes.
3. *For every object X in $\mathcal{N}$ there exists a factorization*
$$R(X) \xrightarrow{p} \mathrm{Path}(R(X)) \xrightarrow{g} R(X) \times R(X)$$
of the diagonal $R(X) \to R(X) \times R(X)$ such that $R(p)$ is a weak equivalence and $R(g)$ is a fibration.

Then the right-induced model structure on $\mathcal{N}$ exists. $\mathcal{N}$ is a combinatorial model category, and if I is a set of generating cofibrations and J is a set of generating trivial cofibrations in $\mathcal{M}$, then $F(I)$ and $F(J)$ are generating sets of (trivial) cofibrations for $\mathcal{N}$.

Proof. Because $\mathcal{N}$ is a locally presentable category, the images of sets of generating (trivial) cofibrations generate weak factorization systems on $\mathcal{N}$. The right-induced model structure on $\mathcal{N}$ then exists by an argument dual to that in the proof of Hess et al. [37, Theorem 2.2.1]. $\square$

The following is inspired by Bourke [14, Theorem 19], which deals with the unenriched case. Note that where Bourke has to resort to a "highly non-functorial path object", simplicial enrichment grants us a highly functorial path object instead.

Proposition 4.38. *Let $\mathcal{M}$ be a combinatorial model* sSet$^+$*-category, and let J be a set of trivial cofibrations in $\mathcal{M}$. Then the model category structure of $\mathcal{M}$ can be transferred to* $\mathrm{Alg}_J(\mathcal{M})$*, endowing* $\mathrm{Alg}_J(\mathcal{M})$ *with the structure of a combinatorial model* sSet$^+$*-category structure. The unit $X \to G(F(X))$ is a trivial cofibration for all X in $\mathcal{M}$, and (F, G) is a Quillen equivalence.*

Proof. Let $J' \supseteq J$ be a superset of trivial cofibrations that is furthermore generating. We then have a triangle of $\mathcal{V}$-adjoint functors

$$\begin{tikzcd} & \mathrm{Alg}_J(\mathcal{M}) \arrow[dl, shift left, "G"] \arrow[dr, shift left, "F'"] & \\ \mathcal{M} \arrow[ur, shift left, "F"] \arrow[rr, shift left, "F''"] & & \mathrm{Alg}_{J'}(\mathcal{M}) \arrow[ul, shift left, "G'"] \arrow[ll, shift left, "G''"] \end{tikzcd}$$

where G' denotes the evident forgetful functor, by the adjoint functor theorem.

We verify the conditions of Lemma 4.37. Set $R(X) = G'(F'(X))$ and $\eta : X \to G'(F'(X)) = R(X)$ as unit of the adjunction. Since every object in the image of G'' is fibrant and $GR = G''F'$, 1 is satisfied, and 2 holds by functoriality of $G'F'$ and naturality of η.

For 3 we take as proposed path object the power $\mathrm{Path}(R(X)) = R(X)^{(\Delta^1)^\sharp}$ in $\mathrm{Alg}_J(\mathcal{M})$. The maps $\partial\Delta^1 \to \Delta^1 \to \Delta^0$ of simplicial sets induce a sequence

$$R(X) \longrightarrow \mathrm{Path}(X) \longrightarrow R(X) \times R(X)$$

which is mapped to

$$G''(F'(X)) \longrightarrow G''(F'(X))^{(\Delta^1)^\sharp} \longrightarrow G''(F'(X)) \times G''(F'(X))$$

under G (which, as a right adjoint, preserves products and powers). $G''(F'(X))$ is fibrant, so this sequence constitutes the canonical path object induced by the simplicial model structure on $\mathcal{M}$, which proves 3. Thus the right-transferred model structure on $\mathrm{Alg}_J(\mathcal{M})$ exists. Since G preserves limits and powers and reflects weak equivalences and fibrations, it follows that $\mathrm{Alg}_J(\mathcal{M})$ satisfies the pullback power axiom, hence has the structure of a model sSet$^+$-structure.

Now let us show that that the unit $\eta_A : A \to G(F(A))$ of the adjunction $F \dashv G$ is indeed a trivial cofibration for all A in $\mathcal{M}$. Let

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\eta_A"'] & X \arrow[d, "f"] \\ G(F(A)) \arrow[r] & Y \end{tikzcd} \tag{4.13}$$

be a commuting square with f a fibration. Denote by (L, R) the sSet^+-enriched functorial factorization system produced by Garner's small object argument, so that $\mathrm{Alg}_J(\mathcal{M})$ is equivalent to the full subcategory of R-algebras with terminal codomain. f has the right lifting property against all trivial cofibrations and in particular against J, hence by Riehl [71, Lemma 13.3.6], f also has the enriched right lifting property against J, and thus admits R-algebra structure. η_A, on the other hand, is the cofree L-coalgebra over $A \to 1$; in particular, it admits L-coalgebra structure. Since L-coalgebras have the left lifting property with respect to R-algebras, it follows that the lifting problem (4.13) has a solution.

To show that (F, G) is a Quillen equivalence, it suffices to show that the components of unit η and counit ε of the adjunction are weak equivalences. We have already shown the components of the unit to be weak equivalences. Let X be an object of $\mathrm{Alg}_J(\mathcal{M})$. By one of the triangle equalities, we have a commuting triangle

$$\begin{tikzcd} & G(F(G(X))) \arrow[dr, "G(\varepsilon_X)"] & \\ G(X) \arrow[ur, "\eta_{G(X)}"] \arrow[rr, "="'] & & G(X), \end{tikzcd}$$

hence by two-out-of-three, $G(\varepsilon_X)$ is a weak equivalence in $\mathcal{M}$. By definition of right transferred model structures, it follows that ε_X is a weak equivalence in $\mathrm{Alg}_J(\mathcal{M})$. □

Strict ∞-categories

From the formalism of algebraically fibrant objects explained in the previous section, we obtain a diagram

$$\begin{tikzcd} \mathrm{sSet}^+ \arrow[r] \arrow[d] & \mathrm{Lex} \arrow[r] \arrow[d] & \mathrm{Lcc} \arrow[d] \\ \mathrm{sCat} \arrow[r] & \mathrm{sLex} \arrow[r] & \mathrm{sLcc} \end{tikzcd}$$

of left Quillen functors which commutes up to isomorphism. sCat is the category of *strict ∞-categories*, sLex the category of *strict lex ∞-categories* and sLcc the category of *strict lcc ∞-categories.* All model categories and functors are simplicial, and for all but Lcc and sLcc the simplicial structure is obtained by change of base along $\mathrm{Core} : \mathrm{sSet}^+ \to \mathrm{sSet}$ of a sSet^+-enrichment. $\mathrm{sCat} = \mathrm{Alg}(\mathrm{sSet}^+)$ and $\mathrm{sLex} = \mathrm{Alg}(\mathrm{Lex})$ are the algebraically fibrant (and sSet^+-enriched) objects of sSet^+ and Lex, respectively. The definition of sLcc is more subtle: Since Lcc is only sSet-enriched, but not sSet^+-enriched, we can instantiate the algebraically fibrant object formalism only for the simplicial enrichment, but then there is no evident right Quillen functor $\mathrm{Alg}(\mathrm{Lcc}) \to \mathrm{Alg}(\mathrm{Lex})$.

Instead sLcc is defined as follows. Let i' be the inclusion of the discrete category $I'_{\mathrm{Lcc}} = (I_{\mathrm{lcc}} \setminus I_{\mathrm{lex}})^\flat \subseteq \mathrm{Lex}$. Then there is a canonical isomorphism of categories $(\mathrm{sSet}^+)^{I_{\mathrm{lcc}}} \cong \mathrm{Lex}^{i'}$. Let W_{lcc} be the family of morphisms in $\mathrm{Lex}^{i'}$ corresponding to the morphisms $j_\Pi^{\Lambda_2^2}, j_\Pi^n, \overrightarrow{\mathbf{f}}$ and $\overrightarrow{\mathbf{f}}$ in $(\mathrm{sSet}^+)^{I_{\mathrm{lcc}}}$ described in Definition 4.13.

Then the model structures of the left Bousfield localization $(W_{\mathrm{lcc}})^{-1}\mathrm{Lex}^{i'}$ corresponds to that of Lcc under the isomorphism $(\mathrm{sSet}^+)^{I_{\mathrm{lcc}}} \cong \mathrm{Lex}^{i'}$. We obtain a diagram

$$\begin{tikzcd}
& \mathrm{Lex} \arrow[r, "F"] \arrow[d, "\flat"'] & \mathrm{sLex} \arrow[d, "\flat"] \\
& \mathrm{Lex}^{i'} \arrow[r, "F^{i'}"] \arrow[d] & \mathrm{sLex}^{Fi'} \arrow[d] \\
\mathrm{Lcc} \arrow[r, "="] & W^{-1}\mathrm{Lex}^{i'} \arrow[r, "W^{-1}(F^{i'})"] & (F^{i'}(W))^{-1}(\mathrm{sLex}^{Fi'})
\end{tikzcd}$$

of left Quillen functors in which the horizontal functors are Quillen equivalences: By Proposition 4.38, F is a Quillen equivalence, which implies by 4.7 that $F^{i'}$ is a Quillen equivalence, which then implies by Hirschhorn [39, Theorem 3.3.20] that $W^{-1}(F^{i'})$ is a Quillen equivalence because every map W is a set of cofibration, hence so is $F^{i'}(W)$. Now sLcc is given by the model category $\mathrm{Alg}((F^{i'}(W))^{-1}\mathrm{sLex}^{Fi'})$ of algebraically fibrant objects with respect to the simplicial (but not marked simplicial) enrichment.

Note that the definition of a strict lcc category is somewhat redundant, in that for some trivial cofibrations more than one canonical lift is available: Let Γ be a strict lcc category and let $j : A \to B$ be a small trivial cofibration in Lex. Then the solution b to a lifting problem

$$\begin{tikzcd}
A^\flat \arrow[r, "a"] \arrow[d, "j^\flat"'] & G(\Gamma) \\
B^\flat \arrow[ur, dashed, "b"']
\end{tikzcd}$$

in Lcc induced by the structure of an algebraically fibrant object of $(F^{i'}(W))^{-1}(\mathrm{sLex}^{Fi'})$ will generally not correspond to the canonical solution b' of the lifting problem

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "j"'] & G(U(G(\Gamma))) \\
B \arrow[ur, dashed, "b'"']
\end{tikzcd}$$

in Lex which is induced by the image of Γ under $\mathrm{sLcc} \xrightarrow{G} \mathrm{sLex}^{Fi'} \xrightarrow{U} \mathrm{sLex}$. In practice, whenever we pick the "canonical" lift for which this ambiguity is possible, either one of the constructions can be used, but the choice must be the same for all Γ so as to be natural in Γ.

Definition 4.39. Let Γ be a strict lcc category. A *type* σ in Γ, denoted by $\Gamma \vdash \sigma$, is an object $\sigma \in U(G(\Gamma))_0$ of the underlying ∞-category of Γ. A *term* s of type σ, denoted by $\Gamma \vdash s : \sigma$, is a morphism $s : t \to \sigma$ in Γ, where $t \in U(G(\Gamma))_0$ is a terminal object given by the canonical lift against the finite trivial cofibration $j^0_{\lim \emptyset}$. (Recall that the domain of $j^0_{\lim \emptyset}$ is the empty sketch and that the domain is the freestanding terminal object.)

If $f : \Gamma \to \Delta$ is a strict lcc functor, then the covariant substitutions $\Gamma \vdash f(s) : f(\sigma)$ are defined by application of f to underlying objects and morphisms of Γ.

Lemma 4.40. *The covariant cwf* sLcc *supports an empty context and context extensions.*

Proof. sLcc is a locally presentable category and in particular cocomplete. In particular, it has an initial object, i.e. an empty context.

Denote by $\{t, x\}$ the sketch given by two freestanding objects t, x, of which t is marked as terminal object, and let $\{k : t \to x\}$ be the freestanding sketch over the edge Δ^1 in which the vertex $\Delta^{\{0\}} = t$ is marked as terminal. There is an evident inclusion $\{t, x\} \subseteq \{k : t \to x\}$. Then terms $\Gamma \vdash s : \sigma$ are in bijection to maps $\{k : t \to x\} \to G(\Gamma)$ which map t to $\mathbf{1}$ and x to σ. Consequently,

$$\begin{tikzcd} F(\{t,x\}) \arrow[r] \arrow[d, "\overline{\langle \mathbf{1}, \sigma \rangle}"] & F(\{k : t \to x\}) \arrow[d] \\ \Gamma \arrow[r, "p"] & \Gamma.\sigma \end{tikzcd} \tag{4.14}$$

defines a context extension by some type $\Gamma \vdash \sigma$. Here $\overline{\langle \mathbf{1}, \sigma \rangle}$ is induced by the adjunction $F \dashv G$ from the map $\langle \mathbf{1}, \sigma \rangle : \{t, x\} \to G(\Gamma)$ that maps t to $\mathbf{1}$ and x to σ. The variable term $\Gamma.\sigma \vdash v : p(\sigma)$ is given by the image of k in $\Gamma.\sigma$. $\square$

Weak identity types

Definition 4.41. A covariant cwf $\mathfrak{C}$ with context extensions *supports weak identity types* if it interprets the following type and term constructors:

$$\frac{\Gamma \vdash s_1 : \sigma \qquad \Gamma \vdash s_2 : \sigma}{\Gamma \vdash \mathrm{Id}\, s_1\, s_2} \qquad\qquad \frac{\Gamma \vdash s : \sigma}{\Gamma \vdash \mathrm{refl}\, s : \mathrm{Id}\, s\, s}$$

$$\frac{\Gamma.(v_1 : \sigma).(v_2 : \sigma).(r : \mathrm{Id}\, v_1\, v_2) \vdash \tau \qquad \Gamma.(u : \sigma) \vdash t : \langle u, u, \mathrm{refl}\, u \rangle(\tau)}{\Gamma.(v_1 : \sigma).(v_2 : \sigma).(r : \mathrm{Id}\, v_1\, v_2) \vdash \mathrm{ind}_{\mathrm{Id}}\, r\, t : \tau}$$

$$\frac{\Gamma.(v_1 : \sigma).(v_2 : \sigma).(h : \mathrm{Id}\, v_1\, v_2) \vdash \tau \qquad \Gamma.(u : \sigma) \vdash t : \langle u, u, \mathrm{refl}\, u \rangle(\tau)}{\Gamma.(u : \sigma) \vdash \mathrm{ev}_{\mathrm{Id}}\, \tau\, t : \mathrm{Id}\, (\langle u, u, \mathrm{refl}\, u \rangle(\mathrm{ind}_{\mathrm{Id}}\, h\, t))\, t}$$

Here $\langle u, u, \mathrm{refl}\, u \rangle : \Gamma.(v_1 : \sigma).(v_2 : \sigma).(h : \mathrm{Id}\, v_1\, v_2) \to \Gamma.(u : \sigma)$ is induced by the projection $p : \Gamma \to \Gamma.(u : \sigma)$ and the mappings $v_1 \mapsto u$, $v_2 \mapsto u$ and $h \mapsto \mathrm{refl}\, u$.

Note that the last term constructor $\mathrm{ev}_{\mathrm{Id}}\,\tau\,t$ is a propositional version of the usual computation rule.

Definition 4.42. The *freestanding pair of parallel morphisms* is given by $P = \Delta^1 \amalg_{\partial\Delta^1} \Delta^1$.

Definition 4.43. Let Γ be a strict lex ∞-category, let $\Gamma \vdash \sigma$ be a type in context Γ and let $\Gamma \vdash s_1 : \sigma$ and $\Gamma \vdash s_2 : \sigma$ be terms of type σ. s_1 and s_2 induce an evident map $\langle s_1, s_2\rangle : P = U(A^0_{\lim P}) \to U(G(\Gamma))$. The *identity type* $\Gamma \vdash \mathrm{Id}\,s_1\,s_2$ is the cone point of the map $\mathrm{Lim}_P = \Delta^0 \star P = U(B^0_{\lim P}) \to U(G(\Gamma))$ defined via the following canonical lift:

$$\begin{tikzcd} A^0_{\lim P} \arrow[r] \arrow[d, "j^0_{\lim P}"'] & G(\Gamma) \\ B^0_{\lim P} \arrow[ur, dashed] & \end{tikzcd}$$

Lemma 4.44. *Let A_{refl} and B_{refl} be the lex sketches defined as follows. $U(A_{\mathrm{refl}}) = \Delta^2$ is the freestanding 2-simplex. The map $\mathrm{Lim}_P = \Delta^0 \star P \twoheadrightarrow \Delta^{\{0\}} \star \Delta^{\{1,2\}} = U(A_{\mathrm{refl}})$ given by collapsing the two non-trivial edges of P onto Δ^1 is marked as equalizer diagram. $U(B_{\mathrm{refl}}) = \Delta^3/\Delta^{\{0,2\}}$ is the quotient of Δ^3 given by collapsing the edge $\Delta^{\{0,2\}}$ to a single vertex point. The markings of B_{refl} are minimal such that the map $U(A_{\mathrm{refl}}) \cong \Delta^{\{1,2,3\}} \twoheadrightarrow U(B_{\mathrm{refl}})$ preserves markings. Then $A_{\mathrm{refl}} \to B_{\mathrm{refl}}$ is a trivial cofibration of lex sketches.*

Proof. Consider the marking-preserving map $U(A^1_{\mathrm{Lim}\,P}) = \partial\Delta^1 \star P \to U(A)$ defined as follows: Its restriction to $\Delta^{\{1\}} \star P$ is the marked cone $\Delta^0 \star P \twoheadrightarrow \Delta^{\{0\}} \star \Delta^{\{1,2\}} = U(A)$ of A. The image of $\Delta^{\{0\}} \star P$ is given by the composite

$$\Delta^0 \star P \twoheadrightarrow \Delta^0 \star \Delta^1 = \Delta^2 \twoheadrightarrow \Delta^{\{1,2\}} \hookrightarrow \Delta^2 = U(A),$$

i.e. the degenerated cone on the edge $\Delta^{\{1,2\}}$. Now $A \to B$ can be obtained as pushout

$$\begin{tikzcd} A^1_{\lim P} \arrow[r] \arrow[d] & B^1_{\lim P} \arrow[d] \\ A \arrow[r] \arrow[ur, phantom, "\ulcorner"] & B \end{tikzcd}$$

of the trivial cofibration $j^1_{\lim P}$. ∎

Definition 4.45. Let $\Gamma \vdash s : \sigma$ be a term. The identity type $\mathrm{Id}\,s\,s$ induces a map $A_{\mathrm{refl}} \to B_{\mathrm{refl}}$. The *reflexivity term* $\Gamma \vdash \mathrm{refl}\,s : \mathrm{Id}\,s\,s$ is the image of $\Delta^{\{0,1\}} \to B_{\mathrm{refl}}$ under the canonical lift $B_{\mathrm{refl}} \to G(\Gamma)$ of $A_{\mathrm{refl}} \to G(\Gamma)$ against the trivial cofibration $A_{\mathrm{refl}} \to B_{\mathrm{refl}}$.

Lemma 4.46. *Let $k_1, k_2 : x \to y$ be a parallel pair of morphisms in a strict ∞-category Γ. Then there are functions $H_1 \rightleftarrows H_2$, where H_1 is the set of diagrams of the form as on the left of*

$$\begin{tikzcd} x & x \arrow[l, "="'] \arrow[r, "="] \arrow[d] & x \arrow[dl, "k_2"] \\ & y \arrow[from=1-1, "k_1"'] & \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "="] \arrow[d, "k_1"'] \arrow[dr] & x \arrow[d, "k_2"] \\ y \arrow[r, "="] & y \end{tikzcd} \tag{4.3}$$

and H_2 is the set of diagrams as on the right. Moreover, the maps $H_1 \rightleftarrows H_2$ can be constructed naturally in (Γ, k_1, k_2).

Proof. Let us construct the map $H_1 \to H_2$. For this it suffices to construct from a 2-simplex as on the left a 2-simplex as on the right:

$$\begin{tikzcd} x \arrow[dr, "k_1"'] & x \arrow[l, "="'] \arrow[d, "\tilde{k}"] \\ & y \end{tikzcd} \qquad \begin{tikzcd} x \arrow[d, "k_1"'] \arrow[dr, "\tilde{k}"] & \\ y \arrow[r, "="'] & y \end{tikzcd}$$

Given a 2-simplex as on the left, consider the $(3,1)$-horn whose 1-skeleton can can be depicted as follows:

$$\begin{tikzcd} & y \arrow[ddr, bend left] & \\ & x \arrow[u, "k_1"] \arrow[dl] \arrow[dr, "\tilde{k}"] & \\ x \arrow[uur, bend left, "k_1"] \arrow[rr, "k_1"'] & & y \end{tikzcd} \tag{4.15}$$

The first face of a filler of this horn is a 2-simplex of the desired shape.

Next let us construct the map $H_2 \to H_1$. Dually to the previous case, it suffices to construct from a 2-simplex as on the right of (4.3) a 2-simplex as on the left of (4.3). Given a simplex as on the right, we obtain a $(3,2)$-horn whose 1-skeleton can again be depicted as (4.15), and the second face of a filler of this $(3,2)$-horn has the desired form. $\square$

Lemma 4.47. *Let A_{idtm} and B_{idtm} be the lex sketches defined as follows. The underlying simplicial set of A_{idtm} is the quotient of $\Delta^0 \star \Delta^0 \amalg_{\Delta^0} \Delta^0 \star P$ in which the start point $\Delta^0 \to \Delta^0 \star \Delta^0$ in the left component of the amalgamation is collapsed onto the start point $\Delta^{\{0\}} \to P \to \Delta^0 \star P$ in the right component. $\mathrm{Lim}_P = \Delta^0 \star P \to U(A_{\mathrm{idtm}})$ is marked as equalizer, and the start vertex $\mathrm{Lim}_\emptyset = \Delta^{\{0\}} \to P$ of the two edges of P is marked as terminal. The underlying simplicial set $U(B_{\mathrm{idtm}})$ of B_{idtm} is the quotient of $\Delta^0 \star \Delta^0 \star P$ in which the edge from the vertex in the left component of the threefold join to the start point of the two edges in P in the third component is collapsed onto a point. There is an evident map $U(A_{\mathrm{idtm}}) \to U(B_{\mathrm{idtm}})$, and the markings of B_{idtm} are minimal such that this map preserves markings. Then $A_{\mathrm{idtm}} \to B_{\mathrm{idtm}}$ is a trivial cofibration of lex sketches.*

Proof. Consider the diagram

$$
\begin{tikzcd}
 & A^2_{\mathrm{Lim}\,\emptyset} \arrow[r, tail, "\sim"] \arrow[d] & B^2_{\mathrm{Lim}\,\emptyset} \arrow[d] \\
 & A^2_{\mathrm{Lim}\,\emptyset}/\Delta^{\{0,2\}} \arrow[r] \arrow[d] & B^2_{\mathrm{Lim}\,\emptyset}/\Delta^{\{0,2\}} \arrow[d] \\
\Delta^0 \star \Delta^0 \amalg_{\Delta^0} \Delta^0 \star P \arrow[r] \arrow[d, tail, "\sim"] & A_{\mathrm{idtm}} \arrow[r] \arrow[d] & A' \arrow[d] \\
\Delta^0 \star \Delta^0 \star P \arrow[r] & B' \arrow[r] & B_{\mathrm{idtm}}
\end{tikzcd}
$$

of pushout squares. Here the map $A^2_{\lim\emptyset}/\Delta^{\{0,2\}}$ is given on the second face by $\Delta^{\{0,1\}} \cong \Delta^0 \star \Delta^0 \to U(A_{\mathrm{idtm}})$, on the zeroth face by $\Delta^{\{1,2\}} \cong \Delta^0 \star \Delta^{\{0\}} \to \Delta^0 \star P \to U(A_{\mathrm{idtm}})$ and on the first face by the degenerated edge on either of the two collapsed points of $U(A_{\mathrm{idtm}})$. B' agrees with B_{idtm} except that the edge that is collapsed in B_{idtm} is a nontrivial loop.

Since trivial cofibrations are stable under pushouts, the lower right square is composed entirely of trivial cofibrations. By stability under composition, then, $A_{\mathrm{idtm}} \to A' \to B_{\mathrm{idtm}}$ is a trivial cofibration. □

Lemma 4.48. *Let $\Gamma \vdash \sigma$ be a type in a lex category Γ. Set $\Gamma.v_1.v_2.h = \Gamma.(v_1 : \sigma).(v_2 : \sigma).(h : \mathrm{Id}\, v_1\, v_2)$ and $\Gamma.u = \Gamma.(u : \sigma)$. Let $r = \langle p_{\Gamma.u}, u, u, \mathrm{refl}\, u\rangle : \Gamma.v_1.v_2.h \to \Gamma.u$ be induced by the reflexivity term on u, and let $i = \langle p_{\Gamma.v_1.v_2.h}, v_1\rangle : \Gamma.u \to \Gamma.v_1.v_2.h$.*

1. *r is a strong homotopy retract with section i, i.e. there exists a map $\phi : \Gamma.v_1.v_2.h \to (\Gamma.v_1.v_2.h)^{\Delta^1}$ such that $\Gamma.u \xrightarrow{i} \Gamma.v_1.v_2.h \xrightarrow{\phi} (\Gamma.v_1.v_2.h)^{\Delta^1}$ factors via the constant homotopy $\Gamma.u \to (\Gamma.u)^{\Delta^1}$ and the two projections of ϕ satisfy the equations $\mathrm{id}^{\Delta^{\{0\}}} \circ \phi = i \circ r$ and $\mathrm{id}^{\Delta^{\{1\}}} \circ \phi = \mathrm{id}$.*

2. *There exists a map $\psi : \Gamma.v_1.v_2.h \to (\Gamma.v_1.v_2.h)^{\Delta^1\times\Delta^1}$ such that $\Gamma.u \xrightarrow{i} \Gamma.v_1.v_2.h \xrightarrow{\psi} (\Gamma.v_1.v_2.h)^{\Delta^1\times\Delta^1}$ factors via the constant cube map $\Gamma.u \to$*

$(\Gamma.u)^{\Delta^1 \times \Delta^1}$ *and the faces of* ψ *satisfy the following equations:*

$$\begin{tikzcd}
\Gamma.v_1.v_2.h \arrow[r, "\psi"] \arrow[d, "\phi"] & (\Gamma.u)^{\Delta^1 \times \Delta^1} \arrow[d] \\
(\Gamma.v_1.v_2.h)^{\Delta^1} \arrow[r, "r^{\Delta^1}"] & (\Gamma.u)^{\Delta^1 \times \Delta^{\{0\}}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma.v_1.v_2.h \arrow[r, "\psi"] \arrow[d, "r"] & (\Gamma.u)^{\Delta^1 \times \Delta^1} \arrow[d] \\
\Gamma.u \arrow[r, "\mathrm{const}"] & (\Gamma.u)^{\Delta^1 \times \Delta^{\{1\}}}
\end{tikzcd}$$

$$\begin{tikzcd}
\Gamma.v_1.v_2.h \arrow[r, "\psi"] \arrow[d, "r"] & (\Gamma.u)^{\Delta^1 \times \Delta^1} \arrow[d] \\
\Gamma.u \arrow[r, "\mathrm{const}"] & (\Gamma.u)^{\Delta^{\{\varepsilon\}} \times \Delta^1}
\end{tikzcd}
\qquad \varepsilon \in \{0, 1\}$$

Moreover, ϕ *and* ψ *can be constructed so as to vary naturally in* (Γ, σ).

Proof. 1. Our strategy is to use the universal property of the context extension $\Gamma.v_1.v_2.h$ with the base map $\Gamma \to \Gamma.v_1.v_2.h \to (\Gamma.v_1.v_2.h)^{\Delta^1}$ induced by the constant homotopy. To define images for the variables, we construct a diagram in $\Delta^1 \otimes B_{\mathrm{idtm}} \to U(G(\Gamma.v_1.v_2.h))$ which can be depicted as follows:

$$\begin{tikzcd}
1 \arrow[r, "="] \arrow[d, "\mathrm{refl}\, v_1"] \arrow[dd, bend right=60, "="'] & 1 \arrow[d, "h"] \arrow[dd, bend left=60, "="] \\
\mathrm{Id}\, v_1\, v_1 \arrow[r] \arrow[d] & \mathrm{Id}\, v_1\, v_2 \arrow[d] \\
1 \arrow[r, "="] \arrow[d, bend right, "v_1"'] \arrow[d, bend left, "v_1"] & 1 \arrow[d, bend right, "v_1"'] \arrow[d, bend left, "v_2"] \\
\sigma \arrow[r, "="] & \sigma
\end{tikzcd} \tag{4.16}$$

Note that some composite edges are omitted. The left-hand side of the diagram, i.e. the image of $\Delta^{\{0\}} \otimes B_{\mathrm{idtm}}$, is given by the definition of the reflexivity term, while the right-hand side is induced by the term h and Lemma 4.47. Now let us define the image of the two lower squares:

$$\begin{tikzcd}
1 \arrow[r, "="] \arrow[d, "v_1"] & 1 \arrow[d, "v_1"] \\
\sigma \arrow[r, "="] & \sigma
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "="] \arrow[d, "v_1"] & 1 \arrow[d, "v_2"] \\
\sigma \arrow[r, "="] & \sigma
\end{tikzcd} \tag{4.17}$$

For the left square we pick the degenerated square at v_1. As for the right square, observe that the image of $\Delta^{\{1\}} \otimes B_{\mathrm{idtm}}$ contains a subdiagram of the form

$$\begin{tikzcd} 1 \arrow[dr, "v_1"'] & 1 \arrow[l, "="'] \arrow[r, "="] \arrow[d] & 1 \arrow[dl, "v_2"] \\ & \sigma. & \end{tikzcd}$$

By Lemma 4.46, we obtain a square as on the right of (4.17) from this.

The two squares define a pair of terms in $(\Gamma.v_1.v_2.h)^{\Delta^1}$. We can thus define $\phi(v_1)$ as the left square and $\phi(v_2)$ as the right square of (4.17). Taking the canonical equalizer in $(\Gamma.v_1.v_2.h)^{\Delta^1}$ of $\phi(v_1$ and $\phi(v_2)$, we obtain a map $\Delta^1 \otimes A_{\mathrm{idtm}} \cup \partial\Delta^1 \otimes B_{\mathrm{idtm}} \to \Gamma.v_1.v_2.h$. Extending this along the pushout product of $\partial\Delta^1 \subseteq \Delta^1$ with $A_{\mathrm{idtm}} \to B_{\mathrm{idtm}}$, we obtain a diagram $\Delta^1 \otimes B_{\mathrm{idtm}} \to U(G(\Gamma.v_1.v_2.h))$ of the form (4.16) as desired.

This diagram contains a square

$$\begin{tikzcd} 1 \arrow[r, "="] \arrow[d, "\mathrm{refl}\, v_1"'] & 1 \arrow[d, "h"] \\ \mathrm{Id}\, v_1\, v_1 \arrow[r] & \mathrm{Id}\, v_1\, v_2 \end{tikzcd}$$

which we take to define $\phi(h)$. By construction, ϕ is indeed a homotopy from ir to the identity on $\Gamma.v_1.v_2.h$.

2. Our conditions on the projections of ψ are equivalent to ψ being a solution to the following lifting problem:

$$\begin{tikzcd} \Gamma.u \arrow[r] \arrow[d] & (\Gamma.u)^{\Delta^1 \times \Delta^1} \arrow[d] \\ \Gamma.v_1.v_2.h \arrow[r] \arrow[ur, dashed, "\psi"] & (\Gamma.u)^{\partial(\Delta^1 \times \Delta^1)} \end{tikzcd} \tag{4.18}$$

Note that context extensions are obtained by pushouts along the cofibration $F(\{t, x\} \to \{k : t \to x\}$ as in (4.14). Thus $i : \Gamma.u \to \Gamma.v_1.v_2.h$ is a cofibration. Because of 1, it is in fact a trivial cofibration. Since $\partial(\Delta^1 \times \Delta^1) \to \Delta^1 \times \Delta^1$ is a monomorphism of simplicial sets, $\Gamma^{\Delta^1 \times \Delta^1} \to \Gamma^{\partial(\Delta^1 \times \Delta^1)}$ is fibration. Thus the lift ψ as in (4.18) exists. We are not done yet, however, because we claimed that ψ can be constructed naturally in (Γ, σ). Fix a choice of ψ for $\Gamma = F(\{x\})$ the free strict lcc ∞-category over a single object x and $\sigma = x$. For arbitrary

$\Gamma \vdash \sigma$ we find a commuting cube which can be depicted as follows:

$$
\begin{tikzcd}
& \Gamma.u \arrow[rr] \arrow[dd] & & (\Gamma.u)^{\Delta^1 \times \Delta^1} \arrow[dd] \\
F(\{x\}).u \arrow[ur] \arrow[rr] \arrow[dd] & & (F(\{x\}).u)^{\Delta^1 \times \Delta^1} \arrow[ur] \arrow[dd] & \\
& \Gamma.v_1.v_2.h \arrow[rr] & & (\Gamma.u)^{\partial(\Delta^1 \times \Delta^1)} \\
F(\{x\}).v_1.v_2.h \arrow[ur] \arrow[rr] & & (F(\{x\}).u)^{\partial(\Delta^1 \times \Delta^1)} \arrow[ur] &
\end{tikzcd}
$$

The left face is a pushout square. From this and the fixed chosen lift for the front face we obtain a lift for the back face. Because the commuting cubes vary naturally in (Γ, σ), so do the lifts for the back faces. □

Proposition 4.49. *The cwf* sLcc *supports weak identity types.*

Proof. The identity type $\Gamma \vdash \mathrm{Id}\, s_1\, s_2$ was constructed in Definition 4.43, and the reflexivity term $\Gamma \vdash \mathrm{refl}\, s : \mathrm{Id}\, s\, s$ was constructed in 4.45.

Recall the retraction-section pair $r : \Gamma.v_1.v_2.h \rightleftarrows \Gamma.u : i$ and the homotopy $\phi : ir \simeq \mathrm{id}$ of Lemma 4.48. Let $\Gamma.v_1.v_2.h \vdash \tau$ and $\Gamma.u \vdash t : r(\tau)$. We define the induction term $\Gamma.v_1.v_2.h \vdash \mathrm{ind}_{\mathrm{Id}}\ \tau\, t : \tau$ as edge $\Delta^{\{0,2\}}$ of the filler for the inner horn

$$
\begin{tikzcd}
& i(r(\tau)) \arrow[dr, "\phi(\tau)"] & \\
1 \arrow[ur, "i(t)"] \arrow[rr, dashed, "\mathrm{ind}_{\mathrm{Id}}\ \tau\, t"'] & & \tau.
\end{tikzcd}
\tag{4.19}
$$

in $\Gamma.v_1.v_2.h$.

Next let us interpret the evaluation term $\Gamma.u \vdash \mathrm{ev}_{\mathrm{Id}}\, \tau\, t : \mathrm{Id}\, r(\mathrm{ind}_{\mathrm{Id}}\, \tau\, t)\, t$. The homotopy ψ of Lemma 4.48 and the definition of the induction term define a diagram $\Delta^1 \otimes \Lambda^2_1 \cup \partial\Delta^1 \otimes \Delta^2 \to U(G(\Gamma.U))$ which can be depicted as follows:

$$
\begin{tikzcd}
& & r(\tau) \arrow[dr, "="] & \\
& 1 \arrow[ur, "t"] \arrow[rr, "t"] & & r(\tau) \\
& r(\tau) \arrow[uur] \arrow[dr, "r(\phi(\tau))"] & & \\
1 \arrow[uur] \arrow[ur, "t"] \arrow[rr, "r(\mathrm{ind}_{\mathrm{Id}}\, \tau\, t)"] & & r(\tau) \arrow[uur] &
\end{tikzcd}
$$

Note that i is a section to r. The left square is degenerated at t, and the right square is $\psi(\tau)$. The back triangle is degenerated while the front triangle is the image of (4.19) under r. From the lower square of a an extension to a diagram $\Delta^1 \otimes \Delta^2 \to U(G(\Gamma.u))$ and Lemma 4.46 we obtain a diagram which can be depicted as follows:

$$\begin{tikzcd}
1 \arrow[dr, "r(\mathrm{ind}_{\mathrm{Id}}\,\tau\,t)"'] & 1 \arrow[l, "="'] \arrow[r, "="] \arrow[d] & 1 \arrow[dl, "t"] \\
& r(\tau) &
\end{tikzcd}$$

This diagram and the identity type $\mathrm{Id}\, r(\mathrm{ind}_{\mathrm{Id}}\,\tau\, t)\, t$ define a map $A^1_{\lim P} \to G(\Gamma.u)$, which we may extend to $B^1_{\lim P}$. We have $U(B^1_{\lim P}) = \Delta^1 \star P$, and the restriction of $B^1_{\lim P} \to G(\Gamma.u)$ to $\Delta^1 \to \Delta^1 \star P$ defines the evaluation term $\Gamma.u \vdash \mathrm{ev}_{\mathrm{Id}}\,\tau\, t : \mathrm{Id}\, r(\mathrm{ind}_{\mathrm{Id}}\,\tau\, t)\, t$. $\square$

Weak product types

Definition 4.50. A covariant cwf $\mathfrak{C}$ with context extensions *supports weak product types* if it interprets the following type and term constructors:

$$\frac{\Gamma \vdash \sigma_1 \qquad \Gamma \vdash \sigma_2}{\Gamma \vdash \mathrm{Prod}\,\sigma_1\,\sigma_2} \qquad\qquad \frac{\Gamma \vdash s_1 : \sigma_1 \qquad \Gamma \vdash s_2 : \sigma_2}{\Gamma \vdash \mathrm{pair}\, s_1\, s_2 : \mathrm{Prod}\,\sigma_1\,\sigma_2}$$

$$\frac{\Gamma.(u : \mathrm{Prod}\,\sigma_1\,\sigma_2) \vdash \tau \qquad \Gamma.(v_1 : \sigma_1).(v_2 : \sigma_2) \vdash t : \langle \mathrm{pair}\, v_1\, v_2\rangle(\tau)}{\Gamma.(u : \mathrm{Prod}\,\sigma_1\,\sigma_2) \vdash \mathrm{ind}_{\mathrm{Prod}}\,\tau\, t : \tau}$$

$$\frac{\Gamma.(u : \mathrm{Prod}\,\sigma_1\,\sigma_2) \vdash \tau \qquad \Gamma.(v_1 : \sigma_1).(v_2 : \sigma_2) \vdash t : \langle \mathrm{pair}\, v_1\, v_2\rangle(\tau)}{\Gamma.(v_1 : \sigma_1).(v_2 : \sigma_2) \vdash \mathrm{ev}_{\mathrm{Prod}}\,\tau\, t : \mathrm{Id}\,(\langle \mathrm{pair}\, v_1\, v_2\rangle(\mathrm{ind}_{\mathrm{Prod}}\,\tau\, t))\, t}$$

Definition 4.51. Let Γ be a strict lex ∞-category, and let $\Gamma \vdash \sigma_1$ and $\Gamma \vdash \sigma_2$ be types in context Γ. σ_1 and σ_2 induce a map $\langle \sigma_1, \sigma_2\rangle : \Delta^0 \amalg \Delta^0 \to G(\Gamma)$.

1. The *product type* $\Gamma \vdash \mathrm{Prod}\,\sigma_1\,\sigma_2$ is the cone point of the map $\mathrm{Lim}_{\Delta^0 \amalg \Delta^0} = \Delta^0 \star (\Delta^0 \amalg \Delta^0) \to G(\Gamma)$ defined via the canonical lift against $j^0_{\lim \Delta^0 \amalg \Delta^0}$.
2. Let $\Gamma \vdash u : \mathrm{Prod}\,\sigma_1\,\sigma_2$ be a term of the product type. The *projection terms* $\Gamma \vdash \pi_1\, u : \sigma_1$ and $\Gamma \vdash \pi_2\, u : \sigma_2$ are defined via the lift of the map $\Delta^0 \star \Delta^0 \cup \Delta^0 \star (\Delta^0 \amalg \Delta^0) \to G(\Gamma)$ induced by u and the definition of the product type against the trivial cofibration $\Delta^0 \star \Delta^0 \star (\Delta^0 \amalg \Delta^0)$:

$$\begin{tikzcd}
& 1 \arrow[dl, dashed, bend right, "\pi_1\, u"'] \arrow[d, "u"] \arrow[dr, dashed, bend left, "\pi_2\, u"] & \\
\sigma_1 & \mathrm{Prod}\,\sigma_1\,\sigma_2 \arrow[l] \arrow[r] & \sigma_2
\end{tikzcd}$$

3. Let $\Gamma \vdash s_1 : \sigma_1$ and $\Gamma \vdash s_2 : \sigma_2$ be terms. The *pair term* $\Gamma \vdash \mathrm{pair}\, s_1\, s_2 : \mathrm{Prod}\,\sigma_1\,\sigma_2$ is defined by the canonical lift of the map $A^1_{\lim \Delta^0 \amalg \Delta^0} \to G(\Gamma)$ induced by s_1, s_2 and the product type $\mathrm{Prod}\,\sigma_1\,\sigma_2$ against the trivial cofibration $j^1_{\lim \Delta^0 \amalg \Delta^0}$:

$$\begin{tikzcd} & 1 \arrow[d, dashed, "\mathrm{pair}\, s_1\, s_2"] \arrow[ddl, bend right, "s_1"'] \arrow[ddr, bend left, "s_2"] & \\ & \mathrm{Prod}\,\sigma_1\,\sigma_2 \arrow[dl] \arrow[dr] & \\ \sigma_1 & & \sigma_2 \end{tikzcd}$$

Lemma 4.52. *Let $\Gamma \vdash \sigma_1$ and $\Gamma \vdash \sigma_2$ be types in a strict lcc ∞-category Γ. Denote by $\Gamma.v_1.v_2 = \Gamma.(v_1 : \sigma_1).(v_2 : \sigma_2)$ the context extension by variables of type σ_1 and σ_2, and denote by $\Gamma.u = \Gamma.(u : \mathrm{Prod}\,\sigma_1\,\sigma_2)$ the context extension by a variable of the product type. Let $f = \langle p_{\Gamma.u}, \pi_1\, u, \pi_1\, u\rangle : \Gamma.v_1.v_2 \to \Gamma.u$ and $g = \langle p_{\Gamma.v_1.v_2}, \mathrm{pair}\, v_1\, v_2\rangle : \Gamma.u \to \Gamma.v_1.v_2$.*

1. *g and f are homotopy inverse relative to Γ. Thus there exists a map $\phi : \Gamma.v_1.v_2 \to (\Gamma.v_1.v_2)^{\Delta^1}$ under Γ such that the diagrams*

$$\begin{tikzcd} \Gamma.v_1.v_2 \arrow[r, "f"] \arrow[d, "\phi"'] & \Gamma.u \arrow[d, "g"] \\ (\Gamma.v_1.v_2)^{\Delta^1} \arrow[r, "\mathrm{id}^{\Delta^{\{0\}}}"] & \Gamma.v_1.v_2 \end{tikzcd} \qquad \begin{tikzcd} \Gamma.v_1.v_2 \arrow[r, "\mathrm{id}"] \arrow[d, "\phi"'] & \Gamma.v_1.v_2 \\ (\Gamma.v_1.v_2)^{\Delta^1} \arrow[ur, "\mathrm{id}^{\Delta^{\{1\}}}"'] & \end{tikzcd}$$

commute, and, dually, there exists a map $\psi : \Gamma.u \to (\Gamma.u)^{\Delta^1}$ under Γ such that the diagrams

$$\begin{tikzcd} \Gamma.u \arrow[r, "g"] \arrow[d, "\psi"'] & \Gamma.v_1.v_2 \arrow[d, "g"] \\ (\Gamma.u)^{\Delta^1} \arrow[r, "\mathrm{id}^{\Delta^{\{0\}}}"] & \Gamma.u \end{tikzcd} \qquad \begin{tikzcd} \Gamma.u \arrow[r, "\mathrm{id}"] \arrow[d, "\psi"'] & \Gamma.u \\ (\Gamma.u)^{\Delta^1} \arrow[ur, "\mathrm{id}^{\Delta^{\{1\}}}"'] & \end{tikzcd}$$

commute.

2. *There exists a map $\epsilon : \Gamma.u \to (\Gamma.v_1.v_2)^{\Delta^2}$ under Γ such that the diagrams*

$$\begin{tikzcd} \Gamma.u \arrow[r, "\psi"] \arrow[d, "\epsilon"'] & (\Gamma.u)^{\Delta^1} \arrow[d, "g^{\Delta^1}"] \\ (\Gamma.v_1.v_2)^{\Delta^2} \arrow[r, "\mathrm{id}^{\Delta^{\{0,1\}}}"] & (\Gamma.v_1.v_2)^{\Delta^1} \end{tikzcd} \qquad \begin{tikzcd} \Gamma.u \arrow[r, "g"] \arrow[d, "\epsilon"'] & \Gamma.v_1.v_2 \arrow[d, "\phi"] \\ (\Gamma.v_1.v_2)^{\Delta^2} \arrow[r, "\mathrm{id}^{\Delta^{\{0,2\}}}"] & (\Gamma.v_1.v_2)^{\Delta^1} \end{tikzcd}$$

$$\begin{tikzcd} \Gamma.u \arrow[r, "g"] \arrow[d, "\epsilon"'] & \Gamma.v_1.v_2 \arrow[d, "\mathrm{const}"] \\ (\Gamma.v_1.v_2)^{\Delta^2} \arrow[r, "\mathrm{id}^{\Delta^{\{1,2\}}}"] & (\Gamma.v_1.v_2)^{\Delta^1} \end{tikzcd}$$

commute. (This is the up-to-homotopy version of one of the triangle equalities.)

Proof. 1. We construct ϕ and ψ using the universal property of the context extensions $\Gamma.v_1.v_2$ and $\Gamma.u$, starting with ϕ. v_1 and v_2 are part of the diagram

$$\begin{tikzcd} & 1 \arrow[d, dashed, "\mathrm{pair}\, v_1\, v_2"] \arrow[ddl, bend right, "v_1"'] \arrow[ddr, bend left, "v_2"] & \\ & \sigma_1 \times \sigma_2 \arrow[dl] \arrow[dr] & \\ \sigma_1 & & \sigma_2 \end{tikzcd} \tag{4.20}$$

while the image of v_1 and v_2 under gf is given by the two projection terms $\pi_i\,(\mathrm{pair}\, v_1\, v_2)$ in

$$\begin{tikzcd} & 1 \arrow[d, dashed, "\mathrm{pair}\, v_1\, v_2"] \arrow[ddl, bend right, "\pi_1\,(\mathrm{pair}\, v_1\, v_2)"'] \arrow[ddr, bend left, "\pi_2\,(\mathrm{pair}\, v_1\, v_2)"] & \\ & \sigma_1 \times \sigma_2 \arrow[dl] \arrow[dr] & \\ \sigma_1 & & \sigma_2. \end{tikzcd} \tag{4.21}$$

The two diagrams agree on their restriction to $\Delta^0 \star \Delta^0 \cup \Delta^0 \star (\Delta^0 \amalg \Delta^0)$. Lifting this data against the pushout product of $\partial\Delta^1 \subseteq \Delta^1$ and $\Delta^0 \star \Delta^0 \cup \Delta^0 \star (\Delta^0 \amalg \Delta^0) \to \Delta^0 \star \Delta^0 \star (\Delta^0 \amalg \Delta^0)$, we obtain a map $\Delta^1 \otimes (\Delta^0 \star \Delta^0 \star \Delta^0 \star (\Delta^0 \amalg \Delta^0)$, i.e. a homotopy between the diagrams (4.21) and (4.20). This homotopy contains homotopies from $g(f(v_1))$ to v_1 and from $g(f(v_2))$ to v_2, which we take to define $\phi(v_1)$ and $\phi(v_2)$.

Next let us construct ψ, i.e. define the image of the variable u. u is part of the diagram $B^1_{\lim \Delta^0 \amalg \Delta^0} \to G(\Gamma)$ which can be depicted as

$$\begin{tikzcd} & 1 \arrow[d, dashed, "u"] \arrow[ddl, bend right, "\pi_1\, u"'] \arrow[ddr, bend left, "\pi_2\, u"] & \\ & \sigma_1 \times \sigma_2 \arrow[dl] \arrow[dr] & \\ \sigma_1 & & \sigma_2 \end{tikzcd} \tag{4.22}$$

and the image of u under fg is given by the term $\mathrm{pair}\,(\pi_1\, u)\,(\pi_2\, u)$ in

$$\begin{tikzcd} & 1 \arrow[d, dashed, "\mathrm{pair}\,(\pi_1\, u)\,(\pi_2\, u)"] \arrow[ddl, bend right, "\pi_1\, u"'] \arrow[ddr, bend left, "\pi_2\, u"] & \\ & \sigma_1 \times \sigma_2 \arrow[dl] \arrow[dr] & \\ \sigma_1 & & \sigma_2. \end{tikzcd}$$

The two diagrams agree on their restriction to $A^1_{\lim \Delta^0 \amalg \Delta^0}$, hence define a map $\partial\Delta^1 \otimes B^1_{\lim \Delta^0 \amalg \Delta^0} \cup \Delta^1 \otimes A^1_{\lim \Delta^0 \amalg \Delta^0} \to G(\Gamma)$. The lift of this map against the pushout product of $\partial\Delta^1 \subseteq \Delta^1$ with $j^1_{\lim \Delta^0 \amalg \Delta^0}$ contains a homotopy from u to $\mathrm{pair}\,(\pi_1\, u)\,(\pi_2\, u)$, which we take as definition of $\psi(u)$.

2. We construct a suitable map $\partial\Delta^2 \otimes B^1_{\lim \Delta^0 \amalg \Delta^0} \cup \Delta^2 \otimes A^1_{\lim \Delta^0 \amalg \Delta^0} \to G(\Gamma.v_1.v_2)$ such that the lift to a map $\Delta^2 \otimes B^1_{\lim \Delta^0 \otimes \Delta^0} \to G(\Gamma.v_1.v_2)$ contains a suitable term of the product of the constant types $\Delta^2 \to \Delta^0 \xrightarrow{\sigma_i}\!\!\to G(\Gamma.v_1.v_2)$ in $(\Gamma.v_1.v_2)^{\Delta^2}$.

The map $\Delta^2 \otimes A^1_{\lim \Delta^0 \amalg \Delta^0} \to G(\Gamma.v_1.v_2)$ we take the constant triangle with vertex

$$\begin{tikzcd}
 & 1 \arrow[ddl, bend right, "\pi_2\,(\mathrm{pair}\,v_1\,v_2)"'] \arrow[ddr, bend left, "\pi_1\,(\mathrm{pair}\,v_1\,v_2)"] & \\
 & \sigma_1 \times \sigma_2 \arrow[dl] \arrow[dr] & \\
\sigma_1 & & \sigma_2
\end{tikzcd}$$

As map $\Delta^{\{0,1\}} \otimes B^1_{\lim \Delta^0 \amalg \Delta^0} \to G(\Gamma.v_1.v_2)$ we take the image under g of the diagram defining ψ; the endpoints of this diagram are given by (4.22) and (4.22). As map $\Delta^{\{1,2\}} \otimes B^1_{\lim \Delta^0 \amalg \Delta^0} \to G(\Gamma.v_1.v_2)$ we take the degenerated line over the diagram

$$\begin{tikzcd}
 & 1 \arrow[d, "\mathrm{pair}\,v_1\,v_2"] \arrow[ddl, bend right, "\pi_2\,(\mathrm{pair}\,v_1\,v_2)"'] \arrow[ddr, bend left, "\pi_1\,(\mathrm{pair}\,v_1\,v_2)"] & \\
 & \sigma_1 \times \sigma_2 \arrow[dl] \arrow[dr] & \\
\sigma_1 & & \sigma_2.
\end{tikzcd}$$

The construction of $\Delta^{\{0,2\}} \otimes B^1_{\lim \Delta^0 \amalg \Delta^0} \to G(\Gamma.v_1.v_2)$ is more involved. Observe first that there exist homotopies from $\phi(v_i)$ to the constant squares on $\pi_i\,(\mathrm{pair}\,v_1\,v_2)$ which can be depicted as

$$\begin{tikzcd}
 & & 1 \arrow[dl, "v_i"'] \arrow[dr, "\pi_i\,(\mathrm{pair}\,v_1\,v_2)"] & \\
 & \sigma_i \arrow[rr, "="'] & & \sigma_i \\
 & 1 \arrow[uur] \arrow[dl, "\pi_i\,(\mathrm{pair}\,v_1\,v_2)"'] \arrow[dr, "\pi_i\,(\mathrm{pair}\,v_1\,v_2)"] & & \\
\sigma_i \arrow[rr, "="'] \arrow[uur, "\phi(v_i)"] & & \sigma_i \arrow[uur] &
\end{tikzcd} \tag{4.23}$$

for $i = 1, 2$.

We now define two maps $\Delta^1 \otimes \Lambda^3_2 \to G(\Gamma.v_1.v_2)$, one for the first and second projection terms, as follows. The restrictions to $\Delta^1 \otimes \Delta^{\{0,1,2\}}$ are given by the two legs of the diagram defining of $\phi(v)$ (with endpoints given by (4.21) and (4.21)). The restrictions to $\Delta^1 \otimes \Delta^{\{0,2,3\}}$ are given by the homotopies (4.23). The restrictions to $\Delta^1 \otimes \Delta^{\{1,2,3\}}$ are degenerated on the projections $\sigma_1 \times \sigma_2 \to \sigma_i$.

The restrictions to $\Delta^1 \otimes \Delta^{\{0,1,3\}}$ of the two extensions $\Delta^1 \otimes \Delta^3 \to G(\Gamma.v_1.v_2)$ define the desired map $\Delta^{\{0,2\}} \otimes B^1_{\lim \Delta^0 \amalg \Delta^0} \to G(\Gamma.v_1.v_2)$. The resulting lift $\Delta^2 \otimes B^1_{\lim \Delta^0 \otimes \Delta^0} \to G(\Gamma.v_1.v_2)$ contains a 2-simplex of terms of product types (i.e. a term in $\Gamma.v_1.v_2^{\Delta^2}$, which we take as definition of $\varepsilon(u)$. $\square$

Proposition 4.53. *The cwf* sLcc *supports weak product types.*

Proof. Product type constructor and pair term constructor were defined in Lemma 4.51.

Let $\Gamma.(u : \mathrm{Prod}\, \sigma_1\, \sigma_2) \vdash \tau$ and $\Gamma.(v_1 : \sigma_1).(v_2 : \sigma_2) \vdash t : \langle \mathrm{pair}\, v_1\, v_2 \rangle(\tau)$. We must construct an induction term $\Gamma.(u : \mathrm{Prod}\, \sigma_1\, \sigma_2) \vdash \mathrm{ind}_{\mathrm{Prod}}\, \tau\, t : \tau$. In the notation of Lemma 4.52, $g = \langle \mathrm{pair}\, v_1\, v_2 \rangle$. We then define the induction term as edge $\Delta^{\{0,2\}}$ of the filler of the following inner 2-horn:

$$\begin{tikzcd}
& f(g(\tau)) \arrow[dr, "\psi(\tau)"] & \\
1 \arrow[ur, "f(t)"] \arrow[rr, dashed, "\mathrm{ind}_{\mathrm{Prod}}\,\tau\,t"'] & & \tau
\end{tikzcd}$$

Next let us define the term

$$\Gamma.(v_1 : \sigma_1).(v_2 : \sigma_2) \vdash \mathrm{ev}_{\mathrm{Prod}}\, \tau\, t : \mathrm{Id}\, (g(\mathrm{ind}_{\mathrm{Prod}}\, \tau\, t))\, t \tag{4.24}$$

corresponding to the evaluation rule. Consider the diagram $\Delta^1 \otimes \Lambda^2_1 \to G(\Gamma.v_1.v_2)$ and its extension to $\Delta^1 \otimes \Delta^2$ which can be depicted as follows:

$$\begin{tikzcd}
& & & g(\tau) \arrow[dr] & \\
& & 1 \arrow[ur, "t"] \arrow[rr, "g(t)"'] & & g(\tau) \\
& g(f(g(\tau))) \arrow[uurr, "\phi(g(\tau))"'] \arrow[drr, "g(\psi(\tau))"] & & & \\
1 \arrow[uurr, "="] \arrow[ur, "g(f(t))"'] \arrow[rrr, "g(\mathrm{ind}_{\mathrm{Prod}}\,\tau\,t)"'] & & & g(\tau) \arrow[uur, "="'] &
\end{tikzcd}$$

Here the left square (the image of $\Delta^1 \otimes \Delta^{\{0,1\}}$ is given by $\phi(t)$, and the right square can be depicted as follows:

$$\begin{tikzcd}
g(f(g(\tau))) \arrow[r, "\phi(g(\tau))"] \arrow[d, "g(\psi(\tau))"'] \arrow[dr, "\phi(g(\tau))"] \arrow[d, phantom, shift left=20, "\varepsilon(\tau)"] & g(\tau) \arrow[d, "="] \\
g(\tau) \arrow[r, "="'] & g(\tau)
\end{tikzcd}$$

The bottom square of the filler is of the form

$$\begin{tikzcd}
1 \arrow[r, "g(\mathrm{ind}_{\mathrm{Prod}}\,\tau\, t)"] \arrow[d, "="'] & g(\tau) \arrow[d, "="] \\
1 \arrow[r, "t"'] & g(\tau)
\end{tikzcd}$$

which induces the desired evaluation term (4.24). $\square$

Weak unit types

Definition 4.54. A covariant cwf $\mathfrak{C}$ with context extensions *supports weak unit types* if it interprets the following type and term constructors:

$$\frac{\Gamma\,\mathrm{Ctx}}{\Gamma \vdash 1} \qquad \frac{\Gamma\,\mathrm{Ctx}}{\Gamma \vdash * : 1} \qquad \frac{\Gamma.(v:1) \vdash \tau \qquad \Gamma \vdash t : \langle * \rangle(\tau)}{\Gamma.(v:1) \vdash \mathrm{ind}_1\, \tau\, t : \tau}$$

$$\frac{\Gamma.(v:1) \vdash \tau \qquad \Gamma \vdash t : \langle * \rangle(\tau)}{\Gamma.(v:1) \vdash \mathrm{ev}_1\, \tau\, t : \mathrm{Id}\,(\langle * \rangle(\mathrm{ind}_1\, \tau\, t))\, t}$$

Definition 4.55. Let Γ be a strict lex ∞-category

1. The *unit type* $\Gamma \vdash 1$ is the cone point of the map $\mathrm{Lim}_{\emptyset} = \Delta^0 \star \emptyset \to G(\Gamma)$ defined via the canonical lift against $j^0_{\lim \emptyset}$.
2. The *unit term* $\Gamma \vdash * : 1$ is given by the degenerated 1-simplex on the unit type 1.

Lemma 4.56. *Let Γ be an lcc ∞-category Γ. Denote by $\Gamma.v = \Gamma.(v:1)$ the context extension by the unit type. Let $f = \langle \mathrm{id}_\Gamma, * \rangle : \Gamma.v \to \Gamma$ and let $g = p_{\Gamma.v}$.*

1. *There exists a strong deformation retract $\phi : \Gamma.v \to \Gamma$ from gf to the identity on $\Gamma.v$ under Γ.*
2. *There exists a map $\varepsilon : \Gamma.v \to \Gamma^{\Delta^2}$ witnessing the commutativity up to homotopy of the diagram*

$$\begin{tikzcd}
 & fgf \arrow[dl, "f^{\Delta^1} \circ \phi"'] \arrow[dr, "="] & \\
f \arrow[rr, "="'] & & f
\end{tikzcd}$$

of functors $\Gamma.v \to \Gamma$.

Proof. 1. Consider the square boundary $\partial(\Delta^1 \to \Delta^1) \to G(\Gamma.v)$ given by

$$\begin{tikzcd} 1 \arrow[r, "="] \arrow[d, "v"'] & 1 \arrow[d, "*"] \\ 1 \arrow[r, "="] & 1. \end{tikzcd}$$

Lifting against the pushout product of $\partial\Delta^1 \subseteq \Delta^1$ and $j^1_{\lim \emptyset}$, this boundary admits a filler $\Delta^1 \times \Delta^1 \to G(\Gamma.v)$. This filler can be identified with a term of the unit type in $(\Gamma.v)^{\Delta^1}$, which we take as definition of $\phi(v)$.

2. Follows by a similar argument as point 2 of Lemma 4.47. $\square$

Proposition 4.57. *The covariant cwf* sLcc *supports weak unit types.*

Proof. By a similar argument as for product types. $\square$

Slices of strict ∞-categories

Note that we have a simplicial adjunction $F : \mathrm{Lcc} \rightleftarrows \mathrm{sLcc} : G$.

Proposition 4.58. *The alternative slice functor* $\mathrm{Lcc}_{\Delta^0/} \to \mathrm{Lcc}$ *extends to a simplicial functor* $\mathrm{sLcc}_{\Delta^0/} \to \mathrm{sLcc}$ *such that*

$$\begin{tikzcd} \mathrm{sLcc}_{\Delta^0/} \arrow[r] \arrow[d] & \mathrm{sLcc} \arrow[d] \\ \mathrm{Lcc}_{\Delta^0/} \arrow[r] & \mathrm{Lcc} \end{tikzcd}$$

commutes. Here $\mathrm{sLcc}_{\Delta^0/}$ *is defined by the pullback*

$$\begin{tikzcd} \mathrm{sLcc}_{\Delta^0/} \arrow[r] \arrow[d] & \mathrm{sLcc} \arrow[d] \\ \mathrm{Lcc}_{\Delta^0/} \arrow[r] & \mathrm{Lcc} \end{tikzcd}$$

or, equivalently, as the coslice category $\mathrm{sLcc}_{F(\Delta^0)/}$.

Proof. Let Γ be a strict lcc category and let $x : \Delta^0 \to G(\Gamma)$ be an object of Γ. To define the 1-categorical lift of the slice functor, we must solve the lifting problems as on the left of

$$\begin{tikzcd} S \otimes A \arrow[r, "a"] \arrow[d, "\mathrm{id}\otimes j"'] & G(\Gamma)^{/x} \\ S \otimes B \arrow[ur, dashed] & \end{tikzcd} \qquad \begin{tikzcd} (S \otimes A)_\triangleright \arrow[r, "a'"] \arrow[d, "(\mathrm{id}\otimes j)_\triangleright"'] & G(\Gamma) \\ (S \otimes B)_\triangleright \arrow[ur, dashed] & \end{tikzcd} \tag{4.25}$$

functorially in simplicial sets S and strict maps in (Γ, x) for all $j \in J$. Taking transposes along the alternative cone/slice adjunction, we see that we can equivalently solve the lifting problem on the right of (4.25). Recall that the lower square and outer rectangle of

$$\begin{tikzcd}
S \otimes \Delta^0 \arrow[r] \arrow[d] & \Delta^0 \arrow[d] \\
S \otimes A_{\triangleright} \arrow[r] \arrow[d, "\mathrm{id} \otimes j_{\triangleright}"'] & (S \otimes A)_{\triangleright} \arrow[d, "(\mathrm{id} \otimes j)_{\triangleright}"] \\
S \otimes B_{\triangleright} \arrow[r] & (S \otimes B)_{\triangleright}
\end{tikzcd}$$

are pushouts, hence the lower square is a pushout.

We can thus solve the lifting problem to the right of (4.25) as follows with lifts against $\mathrm{id}_S \otimes j_{\triangleright}$ (note that $j_{\triangleright} \in J$) and the universal property of pushouts:

$$\begin{tikzcd}
S \otimes A_{\triangleright} \arrow[r] \arrow[d] & (S \otimes A)_{\triangleright} \arrow[r, "a'"] \arrow[d] & G(\Gamma) \\
S \otimes B_{\triangleright} \arrow[r] \arrow[urr, dashed] & (S \otimes B)_{\triangleright} \arrow[ur, dashed] \arrow[ul, phantom, "\ulcorner", very near start] &
\end{tikzcd}$$

The canonical lifts of Γ against $S \otimes j_{\triangleright}$ are functorial in S and Γ, hence so are our lifts defining the strict lcc ∞-category $\Gamma^{/x}$.

To show that slicing extends to a simplicial functor, we must show that the map $\mathrm{Lcc}_{\Delta^0}((G(\Gamma), x), (G(\Gamma'), x')) \to \mathrm{Lcc}(G(\Gamma)^{/x}, G(\Gamma')^{/x'})$ restricts to a map $\mathrm{sLcc}_{\Delta^0}((\Gamma, x), (\Gamma', x')) \to \mathrm{sLcc}(\Gamma^{/x}, \Gamma'^{/x'})$.

n-simplices in $\mathrm{sLcc}_{\Delta^0}((\Gamma, x), (\Gamma', x'))$ can be identified with diagrams

$$\begin{tikzcd}
(\Delta^0)^{\flat} \arrow[r, "x'"] \arrow[d, "x"] & G(\Gamma') \arrow[d, "c"] \\
G(\Gamma) \arrow[r, "f"] & G(\Gamma'^{\Delta^n}),
\end{tikzcd}$$

where c is induced by the unique map $\Delta^n \to \Delta^0$, such that f preserves the canonical lifts of Γ and Γ'^{Δ^n}. The image of the n-simplex f is given by

$$G(\Gamma)^{/x} \xrightarrow{f^{/\Delta^0}} G((\Gamma'^{\Delta^n})^{/cx'}) \xrightarrow{k} G((\Gamma'^{/x'})^{\Delta^n})$$

where the isomorphism k is induced by the Yoneda lemma from the chain of isomorphisms

$$
\begin{aligned}
&\operatorname{Hom}(S, G((\Gamma'^{\Delta^n})^{/cx'})) \\
\cong\ &\operatorname{Hom}_{\widetilde{\Delta^n}\times\widetilde{\Delta^0}/}(\widetilde{\Delta^n} \times S_{\triangleright}, G(\Gamma')) \\
\cong\ &\operatorname{Hom}_{\widetilde{\Delta^0}/}((\widetilde{\Delta^n} \times S)_{\triangleright}, G(\Gamma')) \\
\cong\ &\operatorname{Hom}(S, G((\Gamma'^{/x'})^{\Delta^n})).
\end{aligned}
$$

It thus suffices to show that the map $G((\Gamma^{\Delta^n})^{/cx}) \to G((\Gamma^{/x})^{\Delta^n})$ is strict for all (Γ, x). Let $j : A \to B$ be in J and let $a : S \otimes A \to G((\Gamma^{\Delta^n})^{/cx})$ with corresponding map $a' : S \otimes A \to G((\Gamma^{/x})^{\Delta^n})$. Identifying a and a' with their double transposes along the cone/slice and power/copower adjunctions, we obtain the following diagram:

$$
\begin{tikzcd}
\widetilde{\Delta^n \times S} \times (\Delta^0)^{\flat} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \widetilde{\Delta^n} \times (\Delta^0)^{\flat} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & (\Delta^0)^{\flat} \arrow[d] \arrow[dr, "x"] & \\
\widetilde{\Delta^n \times S} \times A_{\triangleright} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \widetilde{\Delta^n} \times (\tilde{S} \times A)_{\triangleright} \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & (\widetilde{\Delta^n \times S} \times A)_{\triangleright} \arrow[r, "a'"] \arrow[d] & G(\Gamma) \\
\widetilde{\Delta^n \times S} \times B_{\triangleright} \arrow[r] & \widetilde{\Delta^n} \times (\tilde{S} \times B)_{\triangleright} \arrow[r] & (\widetilde{\Delta^n \times S} \times B)_{\triangleright} &
\end{tikzcd}
$$

Here $a = a'u$. Note that $\widetilde{\Delta^n} \times -$ has a right adjoint (exponentiation with $\widetilde{\Delta^n}$) and hence preserves colimits, and that we have $\widetilde{S \times T} \cong \tilde{S} \times \tilde{T}$ for all simplicial sets S and T. Thus the two left squares are indeed pushout squares.

The transpose of the lift for a is induced by the lift of $G(\Gamma)$ for $\widetilde{\Delta^n \times S} \times j_{\triangleright}$ and the left bottom pushout square. The transpose of the image of this lift in $G((\Gamma'^{/x})^{\Delta^n})$ is then induced by the composed pushout square on the right, or, equivalently, by the bottom right pushout square. The transpose of the lift for a', on the other hand, is induced by composing the two pushout squares on the bottom and again the lift of $G(\Gamma)$ for $(\Delta^n \times S) \otimes j_{\triangleright}$. Thus the two lifts agree, hence $G((\Gamma^{\Delta^n})^{/cx}) \to G((\Gamma^{/x})^{\Delta^n})$ is indeed strict. □

Lemma 4.59. *Let $\mathcal{M}$ be a simplicially enriched category which is simplicially complete. Denote by* $\operatorname{Lim}(\mathcal{M}, \mathrm{sSet})$ *the 1-category of simplicial functors which preserve simplicial limits (with respect to the cartesian closed structure of* sSet*) and simplicial natural transformations, and let* $\operatorname{Lim}(\mathcal{M}_0, \mathrm{Set})$ *be the category of limit-preserving functors from the underlying 1-category* $\mathcal{M}_0$ *to* Set*. Then the functor* $\operatorname{Lim}(\mathcal{M}, \mathrm{sSet}) \to \operatorname{Lim}(\mathcal{M}_0, \mathrm{Set})$ *given by* $G \mapsto \operatorname{Hom}(\Delta^0, G(-))$ *is an equivalence.*

Proof. Let $G : \mathcal{M} \to \text{sSet}$ be a simplicial limit-preserving functor. Then for all X in $\mathcal{M}$ and $n \geq 0$ there is an isomorphism

$$\begin{aligned} &\text{Hom}(\Delta^n, G(X)) \\ \cong\ &\text{Hom}(\Delta^0 \times \Delta^n, G(X)) \\ \cong\ &\text{Hom}(\Delta^0, G(X)^{\Delta^n}) \\ \cong\ &\text{Hom}(\Delta^0, G(X^{\Delta^n})), \end{aligned}$$

so G is determined by the points of its images. Conversely, if $G' : \mathcal{M}_0 \to \text{Set}$ is a limit-preserving functor, then setting $\text{Hom}(\Delta^n, G(X)) = G'(X^{\Delta^n})$ defines a simplicial limit preserving functor $\mathcal{M} \to \text{sSet}$: Boundary and degeneracy maps of $G(X)$ are given by the image under G' of maps $X^{\Delta^m} \to X^{\Delta^n}$ induced from maps $[n] \to [m]$. By functoriality of G', the boundary and degeneracy maps of $G(X)$ satisfy the simplicial identities. If $f : X \to Y$ is a map in $\mathcal{M}$ and $k : \Delta^n \to \Delta^m$, then

$$\begin{tikzcd} X^{\Delta^m} \arrow[r, "f^{\Delta^m}"] \arrow[d, "X^k"] & Y^{\Delta^m} \arrow[d, "Y^k"] \\ X^{\Delta^n} \arrow[r, "f^{\Delta^n}"] & Y^{\Delta^n} \end{tikzcd}$$

commutes, hence so does the image of the square under G'. It follows that f induces a natural transformation $G(X) \to G(Y)$. Functoriality of G with respect to 1-cells of $\mathcal{M}$ follows from functoriality of G' applied to maps of the form $X^{\Delta^n} \xrightarrow{f^{\Delta^n}} Y^{\Delta^n} \xrightarrow{g^{\Delta^n}} Z^{\Delta^n}$. Thus G is well-defined as a 1-functor $\mathcal{M}_0 \to \text{sSet}$.

Now let us extend G to a simplicial functor. The isomorphisms

$$\begin{aligned} &\text{Hom}(\Delta^n, G(X^S)) \\ =\ &\text{Hom}(\Delta^0, G((X^S)^{\Delta^n})) \\ \cong\ &\text{Hom}(\Delta^0, G(S \times \Delta^n)) \\ =\ &\text{Hom}(S \times \Delta^n, G(X)) \\ \cong\ &\text{Hom}(\Delta^n, G(X)^S) \end{aligned}$$

induce natural isomorphisms $G(X^S) \cong G(X)^S$ for all X in $\mathcal{M}$ and simplicial sets S. An n-simplex $f : \Delta^n \to \mathcal{M}(X, Y)$ of maps corresponds to a map $\bar{f} : \mathcal{M}(X, Y^{\Delta^n})$, and we define $G(f)$ as the n-simplex of maps from $G(X)$ to (GY) corresponding to the map

$$G(X) \xrightarrow{G(\bar{f})} G(Y^{\Delta^n}) \cong G(Y)^{\Delta^n}.$$

It follows directly from this that G (which we have not proved to be simplicially functorial yet) preserves powers $S \to \mathcal{M}(X^S, X)$. A pair of composable n-simplices f, g in $\mathcal{M}$ corresponds to morphisms $\bar{f} : X \to Y^{\Delta^n}$ and $\bar{g} : Y \to Z^{\Delta^n}$, and the composition gf corresponds to the map

$$\overline{gf} : X \xrightarrow{\bar{f}} Y^{\Delta^n} \xrightarrow{(\bar{g})^{\Delta^n}} (Z^{\Delta^n})^{\Delta^n} \xrightarrow{\cong} Z^{\Delta^n \times \Delta^n} \xrightarrow{Z^\delta} Z^{\Delta^n}$$

where $\delta : \Delta^n \times \Delta^n$ denotes the diagonal. The composition of $G(f)$ and $G(g)$ can be expressed in terms of powers by the analogous map $G(X) \to G(Z)^{\Delta^n}$. Since G preserves all involved operations, it follows that G is functorial also on simplices of maps.

We have shown that $G \mapsto \mathrm{Hom}(\Delta^0, G(-))$ is essentially surjective; let us show next that it is faithful. If $\phi : G \Rightarrow G'$ is a simplicial natural transformation of simplicial limit-preserving functors, then the diagram

$$\begin{tikzcd}
\mathrm{Hom}(\Delta^n, G(X)) \arrow[r, "\phi_X \circ -"] \arrow[d, "\cong"'] & \mathrm{Hom}(\Delta^n, G'(X)) \arrow[d, "\cong"] \\
\mathrm{Hom}(\Delta^0, G(X)^{\Delta^n}) \arrow[r, "(\phi_X)^{\Delta^n} \circ -"] \arrow[d, "\cong"'] & \mathrm{Hom}(\Delta^0, G'(X)^{\Delta^n}) \arrow[d, "\cong"] \\
\mathrm{Hom}(\Delta^0, G(X^{\Delta^n})) \arrow[r, "\phi_{(X^{\Delta^n})} \circ -"] & \mathrm{Hom}(\Delta^0, G'(X^{\Delta^n}))
\end{tikzcd} \tag{4.26}$$

commutes for all X in $\mathcal{M}$; the upper square by the universal property of the power, and the lower by preservation of powers by G and G'. Thus the action of ϕ_X on n-simplices is uniquely determined by the action of $\phi_{X^{\Delta^n}}$ on points.

Finally, let us show fullness. Let $\phi_0 : \mathrm{Hom}(\Delta^0, G(-)) \Rightarrow \mathrm{Hom}(\Delta^0, G'(-))$ be a natural transformation on points. Take the vertical isomorphisms of (4.26) to define $(\phi_X)_n : \mathrm{Hom}(\Delta^n, G(-)) \to \mathrm{Hom}(\Delta^n, G'(-))$. Then $\phi_X : G(X) \to G'(X)$ is a natural transformation because of the naturality squares for ϕ_0 and the maps $X^{\Delta^n} \to X^{\Delta^m}$ induced by maps $\Delta^m \to \Delta^n$. ϕ is a natural 1-categorical transformation because of naturality of ϕ_0 with respect to maps $X^{\Delta^m} \to Y^{\Delta^n}$, and it is simplicially natural because

$$\begin{tikzcd}
G(X) \arrow[r, "G(f)"] \arrow[d, "\phi_X"] & G(Y^{\Delta^n}) \arrow[r, "\cong"] \arrow[d, "\phi_{(Y^{\Delta^n})}"] & G(Y)^{\Delta^n} \arrow[d, "(\phi_Y)^{\Delta^n}"] \\
G(X) \arrow[r, "G'(f)"] & G'(Y^{\Delta^n}) \arrow[r, "\cong"] & G'(Y)^{\Delta^n}
\end{tikzcd}$$

commutes for all $f : X \to Y^{\Delta^n}$ in $\mathcal{M}$. □

The forgetful functors $\mathrm{Lex} \to \mathrm{sSet}^+ \to \mathrm{sSet}$ are faithful. Thus although Lemma 4.59 applies directly only to sSet-valued functors, we also use it frequently to construct natural transformations of sSet^+-valued or Lex-valued functors by showing that the components of natural transformations are valued in marking preserving maps. Note that if $\mathcal{C}, \mathcal{D} \in \mathrm{Ob}\,\mathrm{sSet}^+$ are fibrant, then $\mathrm{sSet}^+(\mathcal{C}, \mathcal{D}) \hookrightarrow \mathrm{sSet}(U(\mathcal{C}), U(\mathcal{D}))$ is an isomorphism, and that for fibrant lex sketches $\mathcal{E}, \mathcal{F}$ the image of $\mathrm{Lex}(\mathcal{E}, \mathcal{F}) \hookrightarrow \mathrm{sSet}(U(\mathcal{E}), U(\mathcal{F}))$ consists of the finite limit-preserving functors.

Lemma 4.60. *Let Γ be a strict lex ∞-category and let $f : \Delta^1 \to G(\Gamma)$. Then the simplicially natural map $G(\Gamma^{\Delta^1})^{/f}_{\mathrm{pb}} \to G(\Gamma)^{/y}$ admits a simplicially natural section.*

Proof. We apply Lemma 4.59 to define a section on points. Note that the functor $(\Gamma, f) \mapsto G(\Gamma^{\Delta^1})^{/f}_{\mathrm{pb}})$ can be expressed by assigning to (Γ, f) first $(G(\Gamma), f) \in \mathrm{Ob}\,\mathrm{Lex}_{\Delta^1/}$, then to $(\mathcal{C}, f)$ the cospan

$$U(\mathcal{C}^{(B^0_{\lim \Lambda^2_2})}) \leftarrow U(\mathcal{C})^{\Delta^1 \times \Delta^1} \to (U(\mathcal{C})^{\Delta^1})^{/f} \tag{4.27}$$

in sSet^+ and then taking the limit over this diagram. (Recall that $B^0_{\lim \Lambda^2_2}$ is the lex sketch given by the underlying simplicial set $\Delta^0 \star \Lambda^2_2$ which is marked as a limit cone. $\mathcal{C}^{B^0_{\lim \Lambda^2_2}}$ is the exponential in Lex.)

All three objects of the cospan (4.27) can be obtained from $(\mathcal{C}, f)$ by application of right adjoints: In the left component, it is the composition

$$(\mathcal{C}, f) \mapsto \mathcal{C} \mapsto \mathcal{C}^{(B^0_{\lim \Lambda^2_2})} \mapsto U(\mathcal{C}^{\Lambda^2_2}).$$

In the middle component, it is the composition

$$(\mathcal{C}, f) \mapsto \mathcal{C} \mapsto U(\mathcal{C}) \mapsto U(\mathcal{C})^{\Delta^1 \times \Delta^1}.$$

In the right component, it is the composition

$$(\mathcal{C}) \mapsto (U(\mathcal{C}), f) \mapsto (U(\mathcal{C})^{\Delta^1}, f) \mapsto (U(\mathcal{C})^{\Delta^1})^{/f}.$$

Note here that exponentiation by Δ^1 defines a right adjoint $\mathrm{sSet}^+_{\Delta^1/} \cong \mathrm{sSet}^+_{\Delta^0 \times \Delta^1/} \to \mathrm{sSet}^+_{\Delta^0/}$. We conclude that $(\Gamma, f) \mapsto (G(\Gamma)^{\Delta^1})^{/f}_{\mathrm{pb}}$ preserves all simplicial limits.

Similarly, $(\Gamma, f : x \to y) \mapsto G(\Gamma)^{/y}$ can be expressed as composition of right adjoints

$$\mathrm{sLcc}_{\Delta^1/} \to \mathrm{sSet}^+_{\Delta^1/} \to \mathrm{sSet}^+_{\Delta^0/} \to \mathrm{sSet}^+.$$

Now let us define the action on points of the section $G(\Gamma)^{/y} \to (G(\Gamma)^{\Delta^1})^{/f}_{\mathrm{pb}}$. A point of $G(\Gamma)^{/y}$ is an edge $g : z \to y$ in $G(\Gamma)$. Together with f this defines a cospan, over which we obtain the canonical pullback square

$$\begin{tikzcd} w \arrow[r, "f'"] \arrow[d, "g'"'] & z \arrow[d, "g"] \\ x \arrow[r, "f"] & y \end{tikzcd}$$

in $G(\Gamma)$ which is induced by the lift against $j^0_{\lim \Lambda^2_2}$. This defines a point of $(G(\Gamma)^{\Delta^1})^{/f}_{\mathrm{pb}}$ over g, and it varies naturally in (Γ, f) because it is defined in terms of canonical pullback squares, which are preserved under morphisms in Γ. ∎

Definition 4.61. Let Γ be a strict lex category and let $f : x \to y$ be a morphism in Γ. The *(canonical) pullback functor* f^* along f is the map

$$f^* : G(\Gamma^{/y}) \to (G(\Gamma)^{\Delta^1})^{/f}_{\mathrm{pb}} \to G(\Gamma^{/x})$$

defined in terms of the map constructed in Lemma 4.60.

Proposition 4.62. *Let Γ be a strict lex category and let $k : \Delta^2 \to U(\Gamma)$ be a 2-simplex in Γ, which we depict as*

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"] & & z. \end{tikzcd}$$

Then $f^* \circ g^*$ and h^* are naturally homotopic, in the sense that there is a map $\phi : U(\Gamma^{/z}) \to U(\Gamma^{/x})^{\Delta^1}$ which varies naturally in (Γ, k) such that

$$\begin{tikzcd} & U(\Gamma^{/z}) \arrow[dl, "f^*\circ g^*"'] \arrow[d, dashed, "\phi"] \arrow[dr, "h^*"] & \\ U(\Gamma^{/x})^{\Delta^{\{0\}}} & U(\Gamma^{/x})^{\Delta^1} \arrow[l] \arrow[r] & U(\Gamma^{/x})^{\Delta^{\{1\}}} \end{tikzcd}$$

commutes.

Proof. By Lemma 4.59, it suffices to define ϕ on points. Let $a : w \to z$ a morphism in Γ corresponding to a point in $G(\Gamma^{/z})$. Then $a'' = f^*(g^*(a))$ is defined in terms of canonical pullback squares

$$\begin{tikzcd} w'' \arrow[r, "f'"] \arrow[d, "a''"] & w' \arrow[r, "g'"] \arrow[d, "a'"] & w \arrow[d, "a"] \\ x \arrow[r, "f"] & y \arrow[r, "g"] & z \end{tikzcd} \tag{4.28}$$

Let $k' : \Delta^2 \to G(\Gamma)$ be the canonical filler of the inner horn $\Lambda^2_1 \to G(\Gamma)$ given by f' and g'. The two pullback squares of diagram (4.28) together with k and k' constitute a map $\partial\Delta^1 \times \Delta^2 \cup \Delta^1 \times \Delta^2 \to G(\Gamma)$, which we may lift canonically to a map $p : \Delta^1 \times \Delta^2 \to G(\Gamma)$ since the inclusion $\partial\Delta^1 \times \Delta^2 \cup \Delta^1 \times \Delta^2 \subseteq \Delta^1 \times \Delta^2$ is a trivial cofibration in the Joyal model structure. The restriction p' of p to $\Delta^1 \times \Delta^{\{0,2\}}$ is a pullback square in $G(\Gamma)$ by the pasting law and in particular marked as such. $a''' = h^*(a)$ is defined by a pullback square

$$\begin{tikzcd} w''' \arrow[r] \arrow[d, "a'''"] & w \arrow[d, "a"] \\ x \arrow[r, "h"] & z \end{tikzcd} \tag{4.29}$$

over a cospan which agrees with the restriction of p' to Λ^2_2. The pullback square (4.29), p' and the degenerated lower cospan thus define a map $\partial\Delta^1 \otimes B^0_{\lim\Lambda^2_2} \cup \Delta^1 \otimes A^0_{\lim\Lambda^2_2} \to U(\Gamma)$, which we lift canonically along the pushout product of the boundary inclusion $\partial\Delta^1 \subseteq \Delta^1$ and the trivial cofibration $j^0_{\lim\Lambda^2_2}$ to a map $p'' : \Delta^1 \otimes B^0_{\lim\Lambda^2_2} \to G(\Gamma)$. $U(B^0)_{\lim\Lambda^2_2} = \Delta^0 \star \Lambda^2_2 \cong \Delta^1 \times \Delta^1$ is a square. The restriction of p' to $\Delta^1 \otimes (\Delta^1 \times \Delta^{\{0\}})$ is an equivalence of a'' with a''' in $G(\Gamma^{/x})^{\Delta^1}$. □

Recall that the nerve functor $N : \mathrm{Cat} \to \mathrm{sSet}$ has a left adjoint $\tau_1 : \mathrm{sSet} \to \mathrm{Cat}$ such that $\tau_1 \dashv N$ is Quillen adjunction of the canonical model structure on Cat with the Joyal model structure. For ∞-categories $\mathcal{C}$ (in the sense of fibrant objects of sSet with the Joyal model structure), $\tau_1(\mathcal{C})$ can be identified with the homotopy category of $\mathcal{C}$: The objects of $\tau_1(\mathcal{C})$ are the vertices of $\mathcal{C}$, and the morphisms of $\tau_1(\mathcal{C})$ are equivalence classes of edges in $\mathcal{C}$ under left homotopy (or, equivalently, right homotopy). Since τ_1 preserves products, the cartesian self-enrichment of sSet induces Cat-enrichment of sSet, i.e. the structure of a 2-category on sSet.

Adjunctions of ∞-categories can be described in terms of this 2-categorical structure. Thus an adjunction consists of ∞-categories $\mathcal{C}, \mathcal{D}$ consists of functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$, an edge η in $\mathrm{sSet}(\mathcal{C}, \mathcal{C})$ from the identity on $\mathcal{C}$ to $G \circ F$, and an edge ε in $\mathrm{sSet}(\mathcal{D}, \mathcal{D})$ from $F \circ G$ to the identity on $\mathcal{D}$ such that the triangle equalities hold the 1-categories $\mathrm{sSet}(\mathcal{C}, \mathcal{C})$ and $\mathrm{sSet}(\mathcal{D}, \mathcal{D})$.

There is an obvious adaptation of the nerve functor that makes it valued in sSet^+ by defining an edge $\Delta^1 \to N(\mathcal{C})$ to be marked if and only if it arises from an isomorphism in $\mathcal{C}$, which gives rise to Quillen adjunction $\mathrm{Cat} \rightleftarrows \mathrm{sSet}^+$ such that the left adjoint preserves finite products. Thus the above description of adjunctions can be expressed in terms of the arising 2-categorical structure on sSet^+.

Lemma 4.63. *Let $f : G(\Gamma) \to G(\Delta)$ be a lex functor of strict lex categories Γ, Δ, and let $g : x \to y$ be an edge in Γ. Then the square*

$$\begin{tikzcd} G(\Gamma^{/y}) \arrow[r, "f^{/y}"] \arrow[d, "g^*"'] \arrow[dr, phantom, "\overset{\phi}{\simeq}"] & G(\Delta^{/f(y)}) \arrow[d, "f(g)^*"] \\ G(\Gamma^{/x}) \arrow[r, "f^{/x}"] & G(\Delta^{/f(x)}) \end{tikzcd} \tag{4.30}$$

commutes up to a canonical natural equivalence ϕ.

Proof. Let $h : z \to y$ be a morphism in Γ corresponding to a point of $\Gamma^{/y}$. Then both $f(g)^*(f^{/y}(h))$ and $f^{/x}(g^*(h))$ are related by a canonical equivalence because they are first projections of pullback squares over the cospan $f(x) \xrightarrow{f(g)} f(y) \xleftarrow{f(h)} f(z)$. We thus obtain a natural homotopy of (4.30) on points, which induces a homotopy in all dimensions by Lemma 4.59. □

Proposition 4.64. *1. Let Γ be a strict lex category and let $f : x \to y$ be a morphism in Γ. Then there exista a left adjoint $\Sigma_f : U(G(\Gamma^{/x})) \to\to U(G(\Gamma^{/y}))$ to f^* and a unit $\eta : \Sigma_f \circ f^* \Rightarrow \mathrm{id} : U(G(\Gamma^{/x})) \to U(G(\Gamma^{/x}))$ which varies naturally in (Γ, f).*

2. Let Γ be a strict lcc category and let $f : x \to y$ be a morphism in Γ. Then there exist a right adjoint $\Pi_f : U(G(\Gamma^{/x})) \to U(G(\Gamma^{/y}))$ to f^* and counit $\varepsilon : f^* \circ \Pi_f \Rightarrow \mathrm{id} : U(G(\Gamma^{/x})) \to U(G(\Gamma^{/x}))$ which vary naturally in (Γ, f).

Proof. By Lemma 4.59, it suffices to construct this data on points. $\square$

Proposition 4.65. *Let $f : U(G(\Gamma)) \to U(G(\Delta))$ be a lex functor of strict lcc categories Γ and Δ. Then the following conditions are equivalent:*

1. *f is an lcc functor.*

2. *Let $g : x \to y$ be a morphism in Γ. Then there exists a natural equivalence ψ in*

$$\begin{tikzcd} U(G(\Gamma^{/x})) \arrow[r, "f^{/x}"] \arrow[d, "\Pi_g"'] \arrow[dr, phantom, "\overset{\psi}{\simeq}"] & U(G(\Delta^{/f(x)})) \arrow[d, "\Pi_{f(g)}"] \\ U(G(\Gamma^{/y})) \arrow[r, "f^{/y}"'] & U(G(\Delta^{/f(y)})) \end{tikzcd}$$

such that $(f^{/y}, f^{/x}, \phi, \psi)$ with ϕ as in (4.30) is a pseudo-map from the adjunction $g^* \dashv \Pi_g$ to $f(g)^* \dashv \Pi_{f(g)}$ in the 2-category sSet^+: Thus if $\varepsilon : g^* \circ \Pi_g \Rightarrow \mathrm{id}$ and $\varepsilon' : f(g)^* \circ \Pi_{f(g)} \Rightarrow \mathrm{id}$ are the counits of the adjunctions, then

$$f(g)^*\psi \circ \phi\Pi_g \circ f^{/x}\varepsilon = \varepsilon' f^{/x}$$

holds in the homotopy category of functors and natural transformations $G(\Gamma^{/y}) \to G(\Delta^{/f(y)})$.

3. *Let $g : x \to y$ be a morphism in Γ. Then there exist right adjoints Π_g to g^* and $\Pi_{f(g)}$ to $f(g)^*$ such that the following Beck-Chevalley condition holds: The mate $f_{/y} \circ \Pi_g \Rightarrow \Pi_{f(g)} \circ f_{/x}$ induced by natural equivalence ϕ of (4.30) and the (co)units of the adjunctions $g^* \dashv \Pi_g$ and $f(g)^* \dashv \Pi_{f(g)}$ is an equivalence.*

Suppose furthermore that $f^{/x}$ has a right adjoint $f'_x : U(G(\Delta^{f(x)})) \to U(G(\Gamma^x))$ for all x. Then f is lcc if and only if the following condition holds:

4. *Let $g : x \to y$ be a morphism in Γ. Then the following Beck-Chevalley condition holds: The mate $f'_x \circ f(g)^* \Rightarrow g^* \circ f'_y$ induced by the natural equivalence ϕ of (4.30) and the (co)units of the adjunctions $g^* \dashv \Pi_g$ and $f(g)^* \dashv \Pi_{f(g)}$ is an equivalence.*

Proof. The equivalence of 2, 3 and 4 (if the left adjoints f'_x exist) follows from general 2-categorical nonsense [48, Exercise 1.8.7 and Lemma 1.8.9 with $\mathbb{B} = \Delta^1$].

Let us show the equivalence of 1 and 2. Assume first that f is an lcc functor, and let us construct a pseudo-map of adjunctions. By Lemma 4.59, constructing ψ on points $h : z \to x$ in $\Gamma^{/x}$ is sufficient. Consider the following diagrams in Δ:

$$\begin{tikzcd}
f(z) \arrow[d, "f(h)"'] & \cdot \arrow[l, "\eta'(f^{/x}(h))"'] \arrow[r] \arrow[d, "f(g)^*(\Pi_{f(g)}(f^{/x}(h)))" description] & \cdot \arrow[d, "\Pi_{f(g)}(f^{/x}(h))"] \\
f(x) & f(x) \arrow[l, "="'] \arrow[r, "f(g)"] & f(y)
\end{tikzcd}$$

$$\begin{tikzcd}
f(z) \arrow[d, "f(h)"'] & \cdot \arrow[l, "f^{/x}(\eta(h))"'] \arrow[r] \arrow[d, "f^{/x}(g^*(\Pi_g(h)))" description] & \cdot \arrow[d, "f^{/y}(\Pi_g(h))"] \\
f(x) & f(x) \arrow[l, "="'] \arrow[r, "f(g)"] & f(y)
\end{tikzcd}$$

Since f is lcc, both diagrams are marked as dependent products via $\widetilde{\mathrm{Pi}}$. We obtain a homotopy relating the two dependent products via the degeneracies on $f(h)$ and $f(g)$ and a lift against the pushout product of $\partial\Delta^1 \subseteq (\Delta^1)^\sharp$ and $\tilde{j}^0_\Pi$. The right face of this homotopy is the equivalence $\psi(h) : \Pi_{f(g)}(f^{/x}(h)) \simeq f^{/y}(\Pi_g(h))$ in $\Delta^{/f(y)}$.

Let us show that the required equation holds. $f(g)^*\psi$ and $\phi\Pi_g$ are defined as left faces of cubes which can be depicted as follows:

$$
\begin{tikzcd}
& \cdot \arrow[rr] \arrow[ddl] \arrow[d, "f(g)^*(\Pi_{f(g)}(f^{/x}(h)))"] & & \cdot \arrow[ddl] \arrow[d, "\Pi_{f(g)}(f^{/x}(h))"] \\
& f(x) \arrow[rr] \arrow[ddl] & & f(y) \arrow[ddl] \\
\cdot \arrow[rr] \arrow[d, "f(g)^*(f^{/y}(\Pi_g(h)))"'] & & \cdot \arrow[d, "f^{/y}(\Pi_g(h))"'] & \\
f(x) \arrow[rr] & & f(y) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \cdot \arrow[rr] \arrow[ddl] \arrow[d, "f(g)^*(f^{/y}(\Pi_g(h)))"] & & \cdot \arrow[ddl] \arrow[d, "f^{/y}(\Pi_g(h))"] \\
& f(x) \arrow[rr] \arrow[ddl] & & f(y) \arrow[ddl] \\
\cdot \arrow[rr] \arrow[d, "f^{/x}(g^*(\Pi_g(h)))"'] & & \cdot \arrow[d, "f^{/y}(\Pi_g(h))"'] & \\
f(x) \arrow[rr] & & f(y) &
\end{tikzcd}
$$

The cubes are composable along the z-axis. Because the bottom faces of both cubes are degenerated, and the right face of the bottom cube is degenerated. It follows that the lower cospan of the cube relating the right squares of the two cubes above is a composition of lower cospans of the two cubes. All faces along the z axis of the cubes are pullback squares. It follows that any composition of the two cubes (4.3) is canonically homotopic to the cube relating the right squares of diagram (4.3).

The compositions $f^{/x}(\varepsilon(h)) \circ (\phi(\Pi_g(h)) \circ f(g)^*\psi)$ in the ∞-category of functors $U(G(\Gamma^{/x})) \to U(G(\Gamma^{/x}))$ can be constructed functorially such that pointwise evaluation corresponds to canonical lifts against $\Lambda^2_1 \subseteq \Delta^2$. Thus if $e_1 = (\phi(\Pi_g(h)) \circ f(g)^*\psi)(h)$, then e_1 is a composition of the left face of the two cubes (4.3). The composition of the two cubes of (4.3) can be constructed such that the left face is equal to e_1, hence e_1 is canonically homotopic to the central square of the homotopy relating the diagrams (4.3). Let e_2 be the composition of e_1 with $f^{/x}(\varepsilon(h))$. Then e_2 is canonically homotopic to the diagonal square in the left cube relating the two diagrams (4.3).

Let $e_1 : f(g)^*(\Pi_{f(g)}(f^{/x}(h))) \to f^{/x}(g^*(\Pi_g(h)))$ be the composition (defined as canonical lift against $\Lambda^2_1 \subseteq \Delta^2$ of $f(g)^*(\psi(h))$ and $\phi(\Pi_g(h))$ in $\Delta^{/f(x)}$, and let e_2 be the composition of e_1 and $f^{/x}(\varepsilon(h))$. Then the composite of the two cubes (4.3) can be chosen such that the left face is e_1, hence there is a canonical homotopy relating e_1 wit the central vertical square e_1' of (4.3) along the z- and y-axes. $\square$

Lemma 4.66. *Let Γ be a strict lex ∞-category, let $f : x \to y$ be a morphism in Γ and let $g : z \to y$ be a morphism corresponding to an object of $\Gamma^{/y}$. Denote by*

$$\begin{tikzcd} z' \arrow[r, "f'"] \arrow[d, "g'"'] & z \arrow[d, "g"] \\ x \arrow[r, "f"] & y \end{tikzcd}$$

the canonical pullback square over (f, g). Then the square

$$\begin{tikzcd} G((\Gamma^{/y})^{/g}) \arrow[r, "(f^*)^{/g}"] \arrow[d, "a"] & G((\Gamma^{/x})^{/g'}) \arrow[d, "b"] \\ G(\Gamma^{/z}) \arrow[r, "(f')^*"] & G(\Gamma^{/z'}) \end{tikzcd}$$

commutes up to homotopy.

Proof. It suffices to construct the required homotopy on points. Thus let

$$\begin{tikzcd} w \arrow[r, "h"] \arrow[d] & z \arrow[d, "g"] \\ y \arrow[r, "="] & y \end{tikzcd} \tag{4.31}$$

be a square in Γ corresponding to an object of $(\Gamma^{/y})^{/g}$. Then $b((f^*)^{/g}(h))$ is given by the left face of the cube

$$\begin{tikzcd} & w' \arrow[rr] \arrow[dl, "h'"'] \arrow[dd] & & w \arrow[dl, "h"] \arrow[dd] \\ z' \arrow[rr, "f'"] \arrow[dd, "f^*(g)"'] & & z \arrow[dd, "g"] & \\ & x \arrow[rr, "f"] \arrow[dl, "="] & & y \arrow[dl, "="] \\ x \arrow[rr, "f"] & & y & \end{tikzcd} \tag{4.32}$$

The back, front and the degenerated bottom faces of this cube are pullback squares. It follows by the pasting law that also the top face is a pullback square. The image of (4.31) under a is the edge $h : w \to z$, and then $(f')^*(h)$ is defined by the canonical pullback square

$$\begin{tikzcd} w'' \arrow[r] \arrow[d, "h''"'] & w \arrow[d, "h"] \\ z' \arrow[r, "f'"] & z \end{tikzcd}$$

The lower cospans of this pullback square and the pullback square given by the top face of (4.32) agree, so we obtain a homotopy of pullback squares restricting to the degenerated homotopy on lower cospans. One face of this homotopy has the form

$$\begin{tikzcd} w'' \arrow[r] \arrow[d, "h''"'] & w' \arrow[d, "h'"] \\ z' \arrow[r, "="] & z \end{tikzcd}$$

which corresponds to the required homotopy $h' \simeq h''$ in $\Gamma^{/z'}$. □

Theorem 4.67 (Beck-Chevalley)**.** *Let* Γ *be a strict lex category, and let*

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "g"'] & y \arrow[d, "h"] \\ z \arrow[r, "k"] & w \end{tikzcd}$$

be a pullback square in Γ*. Then the mate*

$$\begin{tikzcd} G(\Gamma^{/x}) \arrow[d, "\Sigma_g"'] \arrow[r, phantom, "\Leftarrow", shift right=6ex] & G(\Gamma^{/y}) \arrow[l, "f^*"'] \arrow[d, "\Sigma_h"] \\ G(\Gamma^{/z}) & G(\Gamma^{/w}) \arrow[l, "k^*"'] \end{tikzcd}$$

induced by the homotopy $g^* \circ k^* \simeq f^* \circ h^*$ *is an equivalence.*

Proof. It suffices to construct the required homotopy on points. Thus let $\ell : v \to y$ be a morphism in Γ corresponding to a vertex of $G(\Gamma^{/y})$. Then $k^*(\Sigma_h(l))$ is given by

$$\begin{tikzcd} u \arrow[r] \arrow[dd, "k^*(\Sigma_h(\ell))"'] \arrow[dr, phantom, "\lrcorner", very near start] & v \arrow[d, "\ell"] \arrow[dd, bend left, "\Sigma_h(\ell)"] \\ & y \arrow[d, "h"] \\ z \arrow[r, "k"] & w \end{tikzcd} \tag{4.33}$$

while $\Sigma_g(f^*(\ell))$ is given by

$$\begin{tikzcd} u' \arrow[r] \arrow[d, "f^*(\ell)"] \arrow[dd, bend right, "\Sigma_g(f^*(\ell))"'] \arrow[dr, phantom, "\lrcorner", very near start] & v \arrow[d, "\ell"] \\ x \arrow[r, "f"] \arrow[d, "g"] & y \arrow[d, "h"] \\ z \arrow[r, "k"] & w \end{tikzcd}$$

Combining this diagram with the subdiagram of (4.33) given by exclusion of u, we obtain a map $\partial\Delta^1 \times \Delta^2 \amalg_{\partial\Delta^1 \times \Lambda^2_2} \Delta^1 \times \Lambda^2_2 \to G(\Gamma)$, which we extend to a map $\Delta^1 \to \Delta^2 \to G(\Gamma)$. The face $\Delta^1 \times \Delta^{\{0,2\}}$ of this diagram can be depicted as

$$\begin{tikzcd} u' \arrow[r] \arrow[d, "\Sigma_g(f^*(\ell))"'] & v \arrow[d, "\Sigma_h(\ell)"] \\ z \arrow[r, "k"] & w. \end{tikzcd}$$

By the pasting law, it is a pullback square. Since the lower cospan of this diagram agrees with the lower cospan of the pullback square (4.33), we obtain a homotopy of pullback squares. One of the faces of this homotopy can be depicted as

$$\begin{tikzcd} u \arrow[r] \arrow[d, "k^*(\Sigma_h(\ell))"'] & u' \arrow[d, "\Sigma_g(f^*(\ell))"] \\ z \arrow[r, "="] & z, \end{tikzcd}$$

which corresponds the required homotopy in $\Gamma^{/z}$. $\square$

Corollary 4.68. *Let Γ be a strict lcc category and let $f : x \to y$ be a map in Γ. Then the pullback functor $f^* : U(G(\Gamma^{/y})) \to U(G(\Gamma^{/x}))$ is lcc.*

Proof. Combine Proposition 4.65 and Theorem 4.67. $\square$

Proposition 4.69. *Let Γ be a strict lex category and let $t : \Delta^0 \to G(\Gamma)$ be a terminal object. Then the projection $p_t : G(\Gamma^{/t}) \to G(\Gamma)$ has a simplicially natural section $s_t : G(\Gamma) \to G(\Gamma^{/t})$.*

Proof. By Lemma 4.59, it suffices to construct s_t on points, where it can be defined in terms of canonical lifts against $j^1_{\lim \emptyset}$. $\square$

Definition 4.70. If $x : \Delta^0 \to G(\Gamma)$ is an object of a strict lex category Γ, then we denote by x^* the composite $G(\Gamma) \xrightarrow{s_t} G(\Gamma)^{/t} \xrightarrow{(!_x)^*} G(\Gamma)^{/x}$. Here t denotes the canonical terminal object of Γ (the one induced by the lift against $j^0_{\lim \emptyset}$), and $!_x : x \to t$ is the morphism induced by lifts against $j^1_{\lim \emptyset}$.

Definition 4.71. Let $x : \Delta^0 \to G(\Gamma)$ be an object in a strict lex category Γ. The *diagonal* $d_x : s(x) \to x^*(x)$ is the morphism in $G(\Gamma)^{/x}$ from the degenerated edge $x \xrightarrow{=} x$ to $x^*(x)$ defined as follows. Lifts against $j^1_{\lim \emptyset}$ induce a square in $G(\Gamma)$ whose boundary can be depicted as

$$\begin{tikzcd} x \arrow[r, "="] \arrow[d, "="'] \arrow[dr, "!_x"] & x \arrow[d, "s_t(x)"] \\ x \arrow[r, "!_x"] & t \end{tikzcd}$$

where t denotes the canonical terminal object of Γ. This square and the pullback square

$$\begin{tikzcd} x' \arrow[r] \arrow[d, "x^*(x)"'] & x \arrow[d, "s_t(x)"] \\ x \arrow[r, "!_x"] & t. \end{tikzcd}$$

restrict to the same lower cospan, hence define a map $\partial\Delta^1 \otimes B^0_{\lim \Lambda^2_1} \cup \Delta^1 \otimes A^0_{\lim \Lambda^2_1} \to G(\Gamma)$. The canonical lift against the pushout product of $\partial\Delta^1 \subseteq \Delta^1$ and $j^1_{\lim \Lambda^2_2}$ contains a 2-simplex as in the top right half of the square

$$\begin{tikzcd} x \arrow[r] \arrow[d] \arrow[dr] \arrow[d, phantom, shift left=4, "="] & x' \arrow[d, "x^*(x)"] \\ x \arrow[r] & x. \end{tikzcd}$$

This square corresponds to the diagonal morphism d in $\Gamma^{/x}$.

Lemma 4.72. *Let Γ be a strict lex category and let $t_0, t_1 : \Delta^0 \to G(\Gamma)$ be two terminal objects. Let $t : t_0 \to t_1$ be any map. Then*

$$G(\Gamma) \xrightarrow{s_{t_1}} G(\Gamma^{/t_1}) \xrightarrow{t^*} G(\Gamma^{/t_0}) \xrightarrow{p_{t_0}} G(\Gamma)$$

is simplicially natural homotopic to the identity, in the sense that there is a simplicially natural map $\phi : G(\Gamma) \to G(\Gamma)^{\Delta^1}$ with projections $p_{t_0} \circ t^* \circ s_{t_1}$ and id.

Proof. By Lemma 4.59, it suffices to construct ϕ on points. Let $x : \Delta^0 \to G(\Gamma)$. Then $x' = p_{t_0}(t^*(s_{t_1}(x)))$ is defined by a pullback square

$$\begin{tikzcd} x' \arrow[r, "t'"] \arrow[d, "t^*(s_{t_0}(x))"'] & x \arrow[d, "s_{t_1}(x)"] \\ t_0 \arrow[r, "t"] & t_1 \end{tikzcd}$$

in $G(\Gamma)$. Every morphism between terminal objects is an equivalence. It follows that also the square

$$\begin{tikzcd} x \arrow[r, "="] \arrow[d, "s_{t_0}(x)"'] & x \arrow[d, "s_{t_1}(x)"] \\ t_0 \arrow[r, "t"] & t_1 \end{tikzcd}$$

induced by the universal property of t_1 is a pullback square. The two pullback squares agree on the lower cospan, hence they are equivalent. In particular, x' and x are homotopic. $\square$

Definition 4.73. Let $\mathcal{C}$ be a lex category. The ∞-*category of points* $\mathcal{C}_* \in \mathrm{Ob}\,\mathrm{sSet}^+$ is given by $U(\mathcal{C}^T)$, where T is the lex sketch given by $U(\Delta^1)$ with $\Delta^{\{0\}}$ marked as terminal object.

Proposition 4.74. *Let Γ and E be lex categories and let $x : \Delta^0 \to G(\Gamma)$ be an object of Γ. Let S_0 be the set consisting of triples $(\bar{f}, \phi, \psi)$, where $\bar{f} : G(\Gamma_{/x}) \to G(E)$ is a lex functor, $\phi : G(\Gamma) \to G(E)^{\Delta^1}$ is a homotopy such that $G(\Gamma) \xrightarrow{\phi} G(E)^{\Delta^1} \to G(E)^{\Delta^{\{0\}}}$ is equal to $\bar{f} \circ x^*$, and $\psi : \Delta^0 \to (E_*)^{\Delta^1}$ is* *a homotopy of points in E which is compatible with ϕ and the diagonal d_x in that it can be depicted as*

$$\begin{tikzcd} \bar{f}(s(x)) \arrow[r, "d_x"] \arrow[d] & \bar{f}(x^*(x)) \arrow[d, "\psi(x)"] \\ t \arrow[r] & \psi_1(x). \end{tikzcd}$$

for some terminal object t of $G(E)$. Let T_0 be the set consisting of tuples (f,k), where $f : G(\Gamma) \to G(E)$ is an lcc functor and $k : t \to f(x)$ is a morphism in E with t terminal. There is a map $p_0 : S_0 \to T_0$ which assigns to a triple $(\bar{f}, \phi, \psi)$ the tuple (ϕ_1, ψ_1).

The assignments $((\Gamma, x), E) \mapsto S_0$ and $((\Gamma, x), E) \mapsto T_0$ are limit-preserving functors $(\mathrm{sLcc}_{\Delta^0/})^{\mathrm{op}} \times \mathrm{sLcc} \to \mathrm{Set}$, hence by Lemma 4.59, the map p_0 extends to a map $p : S \to T$ of simplicial sets. (Alternatively, S can be described as subobject

$$S \subseteq \mathrm{Lex}_{\simeq}(G(\Gamma_{/x}), G(E)) \times \mathrm{sSet}(\Delta^1, \mathrm{Lex}_{\simeq}(G(\Gamma), G(E))) \times \mathrm{sSet}^+_{\simeq}(\mathcal{T}, G(E))$$

corresponding to an equalizer diagram, and similarly for T.)

Then $p : S \to T$ admits a simplicially natural section $s : T \hookrightarrow S$, and there is a simplicially natural map $h : S \to S^{\Delta^1}$ corresponding to a homotopy $s \circ p \simeq \mathrm{id}$.

Proof. We construct a section s_0 to p_0 on points. Thus let $f : G(\Gamma) \to G(E)$ be an lcc functor and let $k : t_E \to f(x)$ be a morphism in E with t_E a terminal object. We define $\bar{f} : G(\Gamma^{/x}) \to G(E)$ as composition

$$G(\Gamma^{/x}) \xrightarrow{f^{/x}} G(E^{/f(x)}) \xrightarrow{k^*} G(E^{/t_E}) \xrightarrow{p_{t_E}} G(E).$$

Next let us construct $\phi : \bar{f} \circ x^* \simeq f : G(\Gamma) \to G(E)$. We have $x^* = (!_x)^* \circ s_{t_\Gamma}$, where t_Γ denotes the canonical terminal object of Γ. By functoriality of all involved constructions, it follows that $\bar{f} \circ x^* = g \circ f$, where g is the composite

$$g : G(E) \xrightarrow{s_{f(t_\Gamma)}} G(E^{/f(t_\Gamma)}) \xrightarrow{(!_{f(x)})^*} G(E^{/f(x)}) \xrightarrow{k^*} G(E^{/t_E}) \xrightarrow{p_{t_E}} G(E).$$

Denote by

$$\begin{tikzcd} & f(x) \arrow[dr, "!_{f(x)}"] & \\ t_E \arrow[ur, "k"] \arrow[rr, dashed, "t"] & & f(t_\Gamma) \end{tikzcd}$$

the canonical filler for the $(2,1)$-horn given by k and $!_{f(x)}$). By Lemma 4.59 we obtain a homotopy $h_0 : k^* \circ (!_{f(x)})^* \simeq t^*$. By Lemma 4.72, there is a homotopy $h_1 : p_{t_E} \circ t^* \circ s_{f(t_\Gamma)} \simeq \mathrm{id}_{G(E)}$. The homotopies $(p_{t_E})^{\Delta^1} \circ h_0 \circ s_{f(t_\Gamma)}$ and h_1 are maps $G(E) \to G(E)^{\Delta_1}$ whose second and first projections, respectively, agree. We thus obtain a map $G(E) \to G(E)^{\Lambda^2_1}$. A pointwise argument using Lemma 4.59 shows that that $G(E)^{\Delta^2} \to G(E)^{\Lambda^2_1}$ admits a section. We thus obtain a map $G(E) \to G(E)^{\Delta^2}$, and then the composite $G(E) \to G(E)^{\Delta^2} \to G(E)^{\Delta^{\{0,2\}}}$ is a homotopy $g \simeq \mathrm{id}$, which defines the required homotopy $\phi : \bar{f} \circ x^* \simeq f$ by composition with f.

Next let us define ψ. Consider the three squares

$$
\begin{tikzcd}
t_E \arrow[r, "s"] \arrow[d, "s"'] \arrow[dr, phantom, "A"] & f(x) \arrow[d, "f(d)\simeq\langle \mathrm{id},\mathrm{id}\rangle"] & \\
f(x) \arrow[r, "\langle \mathrm{id}, s\circ !_{f(x)}\rangle"] \arrow[d] \arrow[dr, phantom, "B"] & f(x \times x) \arrow[r, "p_2"] \arrow[d, "p_1"] \arrow[dr, phantom, "C"] & f(x) \arrow[d] \\
t_E \arrow[r, "s"] & f(x) \arrow[r] & f(t_\Gamma)
\end{tikzcd}
\tag{4.34}
$$

in $G(E)$: C is the image under f of the pullback square defining $x^*(x)$. Note that $f(x \times x)$ is a (fibre) product in E, but not the canonical one. B is induced by the universal property of the pullback square C via the composite $f(x) \xrightarrow{!_{f(x)}} t_E \xrightarrow{s} f(x)$ on the first projection and $\mathrm{id} : f(x) \to f(x)$ on the second projection. Since the composed square of B and C is a pullback square, it follows by the pasting law that also B is a pullback square. A is induced by the universal property of the pullback square C because the two composites $\langle \mathrm{id}, s\circ !_{f(x)}\rangle \circ s$ and $d \circ s$ are both homotopic to $s : t_E \to f(x)$ after composition with $p_1$and p_2. Because the vertically composed square AB is a pullback square, it follows that also A is a pullback square.

The component ϕ_x is given by a diagram

$$
\begin{tikzcd}
 & & f(x \times x) \arrow[dr] \arrow[dd] & \\
 & \bar{f}(x^*(x)) \arrow[ur] \arrow[rr] \arrow[dd] \arrow[dddl, "\phi_x"'] & & f(x) \arrow[dd] \arrow[dddll] \\
 & & f(x) \arrow[dr] & \\
 & t_E \arrow[ur, "s"] \arrow[rr] \arrow[ddl] & & f(t_\Gamma) \arrow[ddll] \\
f(x) \arrow[r] \arrow[d] & f(x) \arrow[d] & & \\
t_E \arrow[r] & f(t_\Gamma) & &
\end{tikzcd}
$$

in Γ, which includes an edge $\bar{f}(x^*(x)) \to f(x \times x)$. Let us show that there is a triangle of the form

$$\begin{tikzcd}
& \bar{f}(x^*(x)) \arrow[dl, "\phi_x"'] \arrow[dr] & \\
f(x) \arrow[rr, "f(d)"] & & f(x \times x).
\end{tikzcd} \tag{4.35}$$

By the universal property of pullback square C above, it suffices to show that the composites with the two projections p_1 and p_2 are homotopic. For p_1 this follows by composing the homotopies

$$\begin{aligned}
&(\bar{f}(x^*(x)) \to f(x \times x) \xrightarrow{p_1} f(x)) \\
\simeq\ &(\bar{f}(x^*(x)) \to t_E \xrightarrow{s} f(x)) \\
\simeq\ &(\bar{f}(x^*(x)) \xrightarrow{\phi_x} f(x) \to t_E \xrightarrow{s} f(x)) \\
\simeq\ &(\bar{f}(x^*(x)) \xrightarrow{\phi_x} f(x) \xrightarrow{\langle s, !_{t_E} \rangle} f(x \times x) \xrightarrow{p_2} f(x))
\end{aligned}$$

of morphisms in $G(\Gamma)$, and for p_2 it follows by composing the homotopies

$$\begin{aligned}
&(\bar{f}(x^*(x)) \to f(x \times x) \xrightarrow{p_2} f(x)) \\
\simeq\ &(\bar{f}(x^*(x)) \to f(x)) \\
\simeq\ &(\bar{f}(x^*(x)) \xrightarrow{\phi_x} f(x) \xrightarrow{=} f(x)) \\
\simeq\ &(\bar{f}(x^*(x)) \xrightarrow{\phi_x} f(x) \xrightarrow{\langle s, !_{t_E} \rangle} f(x \times x) \xrightarrow{p_1} f(x)).
\end{aligned}$$

The image of the diagonal $d : \mathrm{id}_x \to x^*(x)$ under $\bar{f}$ is given by a diagram

$$\begin{tikzcd}
& \bar{f}(\mathrm{id}_x) \arrow[r] \arrow[dl, "\bar{f}(d)"'] \arrow[ddl, bend left] & f(x) \arrow[dl, "f(d)"'] \arrow[dd, bend left] \\
\bar{f}(x^*(x)) \arrow[r] \arrow[d] & f(x \times x) \arrow[d, "p_1"] & \\
t_E \arrow[r, "s"] & f(x). &
\end{tikzcd} \tag{4.36}$$

The front and back squares are pullbacks, hence by the pasting lemma also the top square is a pullback square. The triangle (4.35) induces an equivalence of lower cospans of the top square of (4.36) with the lower cospan of pullback square C in (4.34) which is compatible with $\phi_x : \bar{\ }(x^*(x)) \to f(x)$. Extending this equivalence of lower cospans to an equivalence of pullback squares, we obtain ψ as one of the faces of the homotopy.

Next let us construct the map $h : S \to S^{\Delta^1}$ corresponding to a homotopy $s \circ p \simeq \mathrm{id}$. As before, it suffices to define h on vertices. Since s is a section to p, we can further reduce to constructing for all pairs (g_0, ϕ_0, ψ_0) and (g_1, ϕ_1, ψ_1)

in $\mathrm{Hom}(\Delta^0, S)$ which map to the same pair (f, k) in $\mathrm{Hom}(\Delta^0, S)$ an edge in S from (g_0, ϕ_0, ψ_0) to (g_1, ϕ_1, ψ_1). The data comprising E and the (g_i, ϕ_i, ψ_i) correspond to objects of a sSet^+-category $\mathfrak{K}$ defined by an equalizer diagram as subcategory of

$$\mathrm{sLex}_{\Delta^0/} \times (\mathrm{Lex}_{G(\Gamma_{/x})/} \times \mathrm{Lex}_{(\Delta^1)^\sharp \times G(\Gamma_{/x})/} \times \mathrm{Lex}_{(\Delta^1)^\sharp \times \mathcal{T}/})^2$$

Coslice categories $\mathcal{M}_{X/}$ of complete enriched categories $\mathcal{M}$ are complete, and the forgetful functors $\mathcal{M}_{X/} \to \mathcal{M}$ preserve limits. Similarly, limits of complete enriched categories are complete, and the projections preserve limits. Thus $\mathfrak{K}$ is a complete sSet^+-category, and the diagrams of solid arrows

$$\begin{tikzcd} & G(E)^{(\Delta^1)^\sharp} \arrow[d] \\ G(\Gamma_{/x}) \arrow[r, "\langle g_0, g_1 \rangle"'] \arrow[ur, dashed, "g"] & G(E)^{\partial\Delta^1} \end{tikzcd} \qquad \begin{tikzcd} & G(E)^{(\Delta^1 \times \Delta^1)^\sharp} \arrow[d] \\ G(E) \arrow[r, "\langle \phi_0, \phi_1 \rangle"'] \arrow[ur, dashed, "\phi"] & G(E)^{((\partial\Delta^1) \times \Delta^1)^\sharp} \end{tikzcd}$$

$$\begin{tikzcd} & G(E)^{(\Delta^1 \times \Delta^1)^\sharp} \arrow[d] \\ \mathcal{T} \arrow[r, "\langle \psi_0, \psi_1 \rangle"'] \arrow[ur, dashed, "\psi"] & G(E)^{((\partial\Delta^1) \times \Delta^1)^\sharp} \end{tikzcd}$$

are natural transformations of limit-preserving functors $\mathfrak{K} \to \mathrm{sSet}$. Thus it suffices by Lemma 4.59 to construct indicated extensions (g, ϕ, ψ) subject to the evident equations on points.

First let us construct g. Let $y : y_0 \to x$ be morphism in Γ, corresponding to a vertex in $\Gamma^{/x}$. Then

$$\begin{tikzcd} y \arrow[r] \arrow[d] & x^*(y) \arrow[d, "x^*(y)"] \\ \mathrm{id}_x \arrow[r, "d"] & x^*(x) \end{tikzcd} \tag{4.37}$$

is a pullback square in Γ. We obtain a diagram

$$\begin{tikzcd} & & & & g_1(y) \arrow[r] \arrow[d] & g_1(x^*(y_0)) \arrow[d] \arrow[dll] \\ & & & f(y_0) \arrow[d] & g_1(\mathrm{id}_x) \arrow[r] \arrow[dll] & g_1(x^*(x)) \arrow[dll] \\ g_0(y) \arrow[r] \arrow[d] & g_0(x^*(y_0)) \arrow[d] \arrow[urr] & t \arrow[r, leftrightarrow, "k"] & f(x) & & \\ g_0(\mathrm{id}_x) \arrow[r] \arrow[urr] & g_0(x^*(x)). \arrow[urr] & & & & \end{tikzcd}$$

Here the front and back faces are given by application of g_0 and g_1 to (4.37). The two vertical squares on the right are given by application of ϕ_0 and ϕ_1 to the morphism $y : y_0 \to x$, and the two horizontal morphisms on the bottom are given by ψ_0 and ψ_1. Since the ψ_i and $\phi_i(y)$ are homotopies, we obtain a homotopy of lower cospans of the front and back pullback squares by inverting ψ_1 and $\phi_1(y)$ and composing. This induces a homotopy of pullback squares, which includes the required edge $g(y) : g_0(y) \to g_1(y)$.

To define ϕ we have to construct squares with boundary

$$\begin{tikzcd}
g_0(x^*(y)) \arrow[r, "g(x^*(y))"] \arrow[d, "\phi_0(y)"'] & g_1(x^*(y)) \arrow[d, "\phi_0(y)"] \\
f(y) \arrow[r] & f(y)
\end{tikzcd}$$

(the bottom edge can be arbitrary). This can be done by lifting the upper horseshoe against the trivial cofibration given by the pushout product of the trivial cofibration $\Delta^{\{0\}} = (\Delta^{\{0\}})^\sharp \to (\Delta^1)^\sharp$ and the cofibration $\partial\Delta^1 \to \Delta^1$.

Similarly, ψ has to be cube with boundary

$$\begin{tikzcd}
& g_1(\mathrm{id}_x) \arrow[rr] \arrow[d] \arrow[drr, phantom, "\psi_1"] & & g_1(x^*(x)) \arrow[d] \\
& t \arrow[rr] & & f(x) \\
g_0(\mathrm{id}_x) \arrow[rr] \arrow[d] \arrow[uur] \arrow[drr, phantom, "\psi_0"] & & g_0(x^*(x)) \arrow[d] \arrow[uur] \arrow[ur, phantom, "\phi(x)"] & \\
t \arrow[rr] \arrow[uur] & & f(x). \arrow[uur] &
\end{tikzcd}$$

Thus front and back face of this cube have to agree with ψ_0 and ψ_1, the right face has to be $\phi(x)$), and the top face has to $g(x)$. All other faces can be arbitrary. The left face can be constructed using the universal property of the terminal object. This data can now be lifted against the pushout product of the trivial cofibration $\Delta^{\{0\}} \to (\Delta^1)^\sharp$ and the boundary inclusion $\partial(\Delta^1 \times \Delta^1) \to \Delta^1 \times \Delta^1$ of the square. $\square$

4.4 Algebraically cofibrant strict ∞-categories

Strictification

Fix a combinatorial and simplicial model category $\mathcal{M}$. Let J be a set of representatives of isomorphism classes of κ-small trivial cofibrations for some κ such that J is generating and closed under tensors by finite simplicial sets. We denote by $\mathrm{Alg}(\mathcal{M}) = \mathrm{Alg}_J(\mathcal{M})$ the combinatorial and simplicial model category of algebraically fibrant objects with respect to J.

Lemma 4.75. *Let $i : S \to T$ be a cofibration of finite simplicial sets S and T, and let X be an algebraically fibrant object of $\mathcal{M}$. Then the map $G(X^T) \to G(X^S)$ induced by i carries the structure of an algebraic fibration in $\mathcal{M}$ which varies naturally in maps $f : X \to Y$ of algebraically fibrant objects.*

Proof. Let $j : A \to B$ be a trivial cofibration in J. Then solutions of an (enriched) lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & G(X^T) \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & G(X^S) \end{tikzcd}$$

correspond to solutions of the lifting problem

$$\begin{tikzcd} T \otimes A \amalg_{S \otimes A} S \otimes B \arrow[r] \arrow[d, "i \square j"'] & G(X) \\ S \otimes B. \arrow[ur, dashed] & \end{tikzcd}$$

Lifting problems have canonical solutions since J is closed under tensors by finite simplicial sets, so that the map $i \square j$ is isomorphic to a map in J. $\square$

Lemma 4.76. *Let $\mathcal{M}$ be a simplicial and combinatorial model category. Let $X \in \operatorname{Ob}\mathcal{M}$ and let $Y \in \operatorname{Alg}(\mathcal{M})$. Then the map*

$$\mathcal{M}(G(F(X)), G(Y)) \to \mathcal{M}(X, G(Y)) \tag{4.38}$$

induced by the unit $\eta_X : X \to G(F(X))$ carries the structure of an algebraic trivial Kan fibration such that maps $f : X' \to X'$ in $\mathcal{M}$ and maps $g : Y \to Y'$ in $\operatorname{Alg}(\mathcal{M})$ induce maps of algebraic trivial Kan fibrations.

Proof. Lifts of a boundary inclusion $\partial\Delta^n \subseteq \Delta^n$ against the map (4.38) correspond to lifts

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\eta_X"'] & G(Y^{\Delta^n}) \arrow[d] \\ G(F(X)) \arrow[r] \arrow[ur, dashed] & G(Y^{\partial\Delta^n}) \end{tikzcd} \tag{4.39}$$

in $\mathcal{M}$. The vertical morphism on the right-hand side carries the structure of an algebraic fibration which varies functorially in morphisms in Y by Lemma 4.75. The unit η_X on the left-hand side is the left part of the factorization of $X \to 1$ into an algebraic trivial cofibration followed by an algebraic cofibration; in particular, η_X carries the structure of an algebraic trivial cofibration. It follows that lifting problems as in (4.39) admits canonical solutions. $\square$

Lemma 4.77. *The composition*

$$G(F(G(\Gamma))) \xrightarrow{G(\varepsilon)} G(\Gamma) \xrightarrow{\eta_{G(\Gamma)}} G(F(G(\Gamma))$$

is homotopic to the identity on $G(F(G(\Gamma)))$. There is a family of homotopies $\psi_\Gamma : G(\varepsilon) \circ \eta_{G(\Gamma)} \simeq \mathrm{id}$ which is natural in Γ in the sense that if $f : \Gamma \to \Gamma'$, then

$$\psi_{\Gamma'} \circ G(F(G(f))) = G(F(G(f)) \circ \psi_\Gamma.$$

Proof. Apply Lemma 4.76 to the square

$$\begin{tikzcd}
\partial\Delta^1 \arrow[r] \arrow[d] & \mathrm{Lcc}(G(F(G(\Gamma))), G(F(G(\Gamma)))) \arrow[d, "- \circ \eta_{G(\Gamma)}"] \\
\Delta^1 \arrow[r] & \mathrm{Lcc}(G(\Gamma), G(F(G(\Gamma))))
\end{tikzcd}$$

where the top morphism is given $G(\varepsilon) \circ \eta_{G(\Gamma)}$ and the identity on the two vertices of $\partial\Delta^1 = \Delta^0 \amalg \Delta^0$, and the bottom arrow is the identity homotopy on $\eta_{G(\Gamma)}$. $\square$

Let $C = FG : \mathrm{sLcc} \to \mathrm{Lcc} \to \mathrm{sLcc}$ be the comonad given by G and F.

Lemma 4.78. *Let $\lambda : \Gamma \to C(\Gamma)$ be a coalgebra. Then there is a homotopy*

$$\phi_\lambda : \eta_{G(\Gamma)} \simeq G(\lambda) : G(\Gamma) \to G(F(G(\Gamma))).$$

ϕ_λ can be constructed such that it is compatible with coalgebra morphisms, i.e. so that if $f : (\Gamma, \lambda) \to (\Gamma', \lambda')$, then $\phi_\lambda \circ G(F(G(f))) = G(f) \circ \phi_{\Gamma'}$.

Proof. Set $\phi_\Gamma = G(\lambda) \circ \psi_\Gamma$. Then the domain of ϕ_Γ is $G(\lambda) \circ G(\varepsilon) \circ \eta_{G(\Gamma)} = \eta_{G(\Gamma)}$, and its codomain is $G(\lambda)$. $\square$

Lemma 4.79 (Strictificaton)**.** *Let $\lambda : \Gamma \to C(\Gamma)$ be a C-coalgebra and let E be a strict lcc category. Then the inclusion*

$$i : \mathrm{sLcc}(\Gamma, E) \hookrightarrow \mathrm{Lcc}(G(\Gamma), G(E))$$

is the inclusion of a deformation retract $r : \mathrm{Lcc}(G(\Gamma), G(E)) \twoheadrightarrow \mathrm{sLcc}(\Gamma, E), h : \mathrm{id} \Rightarrow rs$. r and h are compatible with C-coalgebra morphisms in (Γ, λ) and strict lcc functors in E.

Proof. r is defined as composite

$$\mathrm{Lcc}(G(\Gamma), G(E)) \xrightarrow{\sim} \mathrm{sLcc}(F(G(\Gamma)), E) \xrightarrow{- \circ \lambda} \mathrm{sLcc}(\Gamma, E).$$

If f is a simplex of strict lcc functors $\Gamma \to E$, then the image of $G(f)$ in $\mathrm{sLcc}(F(G(\Gamma)), E)$ is $f\varepsilon$, hence $r(f) = f\varepsilon\lambda = f$.

The composition $i \circ r$ is given by the composition

$$\begin{aligned} & \mathrm{Lcc}(G(\Gamma), G(E)) \\ \xrightarrow{\sim} \; & \mathrm{sLcc}(F(G(\Gamma)), E) \\ \rightarrow \; & \mathrm{Lcc}(G(F(G(\Gamma))), G(E)) \\ \xrightarrow{- \circ G(\lambda)} \; & \mathrm{Lcc}(G(\Gamma), G(\Delta)) \end{aligned}$$

Here if we replace the last map $- \circ G(\lambda)$ by $- \circ \eta_{G(\lambda)}$, then we obtain the identity. Thus the homotopy $\phi_\Gamma : \eta_{G(\Gamma)} \simeq G(\lambda)$ of Lemma 4.78 induces a homotopy $\mathrm{id} \simeq r \circ i$. $\square$

Base types and context extensions

Definition 4.80. Let $\lambda : \Gamma \to C(\Gamma)$ be a coalgebra. A *base type* in (Γ, λ) is an object $\sigma \in \Gamma_0$ such that $\eta(\sigma) = \lambda(\sigma)$. A *base term* of type σ in (Γ, λ) is a vertex $s : t \to \sigma$ with t terminal such that $\eta(s) = \lambda(s)$.

Remark 4.81. Types in a strict lcc category Γ can be equated with maps $\sigma : F(\{x\}) \to \Gamma$, where $\{x\}$ denotes the minimally marked lcc sketch given by a single object $\{x\}$. If $\lambda : \Gamma \to C(\Gamma)$ is coalgebra structure on Γ, then σ is a base type of (Γ, λ) if and only if the square

$$\begin{tikzcd} F(\{x\}) \arrow[r, "\sigma"] \arrow[d, "F(\eta)"] & \Gamma \arrow[d] \\ F(G(F(\{x\}))) \arrow[r, "C(\sigma)"] & C(\Gamma) \end{tikzcd}$$

commutes. Note that $F(\eta)$ defines coalgebra structure on $F(\{x\})$, thus the base types can be understood as those types which are detected by coalgebra morphisms.

Similarly, a map $s : t \to \sigma$ in Γ is a base term if and only if the square

$$\begin{tikzcd} F(\{k : t \to x\}) \arrow[r, "s"] \arrow[d, "F(\eta)"] & \Gamma \arrow[d] \\ F(G(F(\{k : t \to x\}))) \arrow[r, "C(s)"] & C(\Gamma) \end{tikzcd}$$

Here $\{k : t \to x\}$ is the lcc sketch on the free-standing edge Δ^1 in which object $t = \Delta^{\{0\}}$ is marked as terminal.

Proposition 4.82. *Let $\lambda : \Gamma \to C(\Gamma)$ be a coalgebra and let σ be a base type in Γ. Then there exists coalgebra structure $\lambda : \Gamma.\sigma \to C(\Gamma.\sigma)$ such that the following hold:*

- $p : (\Gamma, \lambda) \to (\Gamma.\sigma, \lambda.\sigma)$ *is a coalgebra morphisms.*
- *The variable term* $v : t \to \Gamma'$ *is a base term.*
- *If* $f : (\Gamma, \lambda) \to (\Gamma', \lambda')$ *is a coalgebra morphisms and* $\Gamma' \vdash s : f(\sigma)$ *is a base term, then the induced map* $\langle f, s\rangle : (\Gamma.\sigma, \lambda.\sigma) \to (\Gamma', \lambda')$ *is a coalgebra morphism.*

Proof. Note that the forgetful functor Coa sLcc $\to$ sLcc commutes with colimits and consider the pushout of the following span in Coa sLcc:

$$\begin{tikzcd} F(\{k : t \to x\}) \arrow[d, "F(\eta)"] & F(\{x\}) \arrow[l] \arrow[r, "\sigma"] \arrow[d, "F(\eta)"] & \Gamma \arrow[d, "\lambda"] \\ F(G(F(\{k : t \to x\}))) & F(G(F(\{x\}))) \arrow[l] \arrow[r, "C(\sigma)"] & C(\Gamma). \end{tikzcd} \tag{4.40}$$

□

Remark 4.83. It is unlikely that Coa sLcc admits context extensions for arbitrary types.

Proposition 4.84. *Let* $\lambda : \Gamma \to C(\Gamma)$ *be a coalgebra and let* $\Gamma \vdash \sigma$ *be a base type. Then there is a canonical equivalence*

$$a : G(\Gamma.\sigma) \rightleftarrows G(\Gamma^{/\sigma}) : b$$

$$\alpha : b \circ a \simeq \mathrm{id} \qquad\qquad \beta : a \circ b \simeq \mathrm{id}$$

which is preserved up to equality under coalgebra morphisms $f : (\Gamma, \lambda) \to (\Gamma', \lambda')$ *in the sense that all squares in*

$$\begin{tikzcd} G(\Gamma.\sigma)^{\Delta^1} \arrow[d] & G(\Gamma.\sigma) \arrow[l, "\alpha"'] \arrow[r, shift left, "a"] \arrow[d] & G(\Gamma^{/\sigma}) \arrow[l, shift left, "b"] \arrow[r, "\beta"] \arrow[d] & G(\Gamma^{/\sigma})^{\Delta^1} \arrow[d] \\ G(\Gamma'.f(\sigma))^{\Delta^1} & G(\Gamma'.f(\sigma)) \arrow[l, "\alpha'"'] \arrow[r, shift left, "a'"] & G((\Gamma')^{/f(\sigma)}) \arrow[l, shift left, "b'"] \arrow[r, "\beta'"] & G((\Gamma')^{/f(\sigma)})^{\Delta^1} \end{tikzcd}$$

commute.

Proof. By the Strictification Lemma 4.79, there exists a strict lcc functor $(\sigma^*)^s : \Gamma \to \Gamma^{/\sigma}$ and a homotopy $\zeta : \sigma^* \simeq G((\sigma^*)^s)$. Let d^s be the canonical composition of the $(2, 1)$-horn

$$\mathrm{id}_\sigma \xrightarrow{d} \sigma^*(\sigma) \xrightarrow{\zeta(\sigma)} (\sigma^*)^s(\sigma)$$

in $\Gamma^{/\sigma}$. Then $a = G(\langle(\sigma^*)^s, d^s\rangle)$. b is defined from Proposition 4.74 from the coprojection $G(p) : G(\Gamma) \to G(\Gamma.\sigma)$ and the variable $\Gamma \vdash v : p(\sigma)$.

To obtain an equivalence $\beta : a \circ b \simeq \mathrm{id} : G(\Gamma^{/\sigma}) \to G(\Gamma^{/\sigma})$, it suffices by Proposition 4.74 to construct an equivalence $\phi : a \circ b \circ \sigma^* \simeq \sigma^* : G(\Gamma) \to G(\Gamma^{/\sigma})$ and an equivalence

$$\begin{tikzcd} a(b(\mathrm{id}_\sigma)) \arrow[r, "a(b(d))"] \arrow[d] \arrow[dr, phantom, "\psi"] & a(b(\sigma^*(\sigma))) \arrow[d, "\phi(\sigma)"] \\ \mathrm{id}_\sigma \arrow[r, "d"] & \sigma^*(\sigma). \end{tikzcd} \tag{4.41}$$

By definition of b, there is an equivalence $b \circ \sigma^* \simeq p$, and by definition of a, we have $a \circ p = (\sigma^*)^s \simeq \sigma^*$. Composing these equivalences, we obtain an equivalence $\phi : a \circ b \circ \sigma^* \simeq \sigma^*$ as required. By definition of b, there is an equivalence

$$\begin{tikzcd} b(\mathrm{id}_\sigma) \arrow[r, "b(d)"] \arrow[d] & b(\sigma^*(\sigma)) \arrow[d] \\ t_v \arrow[r, "v"] & p(\sigma) \end{tikzcd}$$

of pointed objects in $G(\Gamma.\sigma)$. Combining the image of this equivalence and the definition of $a(v)$, we obtain a diagram

$$\begin{tikzcd} a(b(\mathrm{id}_\sigma)) \arrow[r, "a(b(d))"] \arrow[d] & a(b(\sigma^*(\sigma))) \arrow[d] \\ a(t_v) \arrow[r, "a(v)=d^s"] \arrow[d] & a(p(\sigma)) \arrow[d] \\ \mathrm{id}_\sigma \arrow[r, "d"] & \sigma^*(\sigma) \end{tikzcd} \tag{4.42}$$

The value of ϕ at σ is a composition of the two right vertical maps. Thus the two squares (4.42) can be composed such that the composition of the right two vertical maps agrees with $\phi(\sigma)$, so that the composition is of the form (4.41).

Recall that $\lambda.\sigma : \Gamma.\sigma \to F(G(\Gamma.\sigma))$ is defined as pushout of the cospan (4.40). The underlying pushout diagram in sLcc has a universal property with respect to 1-simplices of maps, and the strictification operator due to Lemma 4.79 is compatible with coalgebra morphisms. Thus we can successively reduce the construction of α to the construction of

1. an equivalence $(b \circ a)^s \simeq \mathrm{id}^s = \mathrm{id} : \Gamma.\sigma \to \Gamma.\sigma$;
2. an equivalence $\phi : (b \circ a)^s \circ p \simeq p : \Gamma \to \Gamma.\sigma$ and an equivalence $\psi : (b \circ a)^s(v) \simeq v$ in $G(\Gamma.\sigma)_*$ which is compatible with $\phi(\sigma)$; and finally
3. an equivalence $\phi : b \circ a \circ G(p) \simeq G(p) : G(\Gamma) \to G(\Gamma.\sigma)$ and an equivalence $\psi : (b \circ a)(v) \simeq v$ in $G(\Gamma.\sigma)_*$ which is compatible with $\phi(\sigma)$.

Let us construct the data ϕ, ψ as in point 3. By definition of a and b, we have $a \circ G(p) = G((\sigma^*)^s \simeq \sigma^*$, and $b \circ \sigma^* \simeq p$, which defines ϕ. There are squares

$$\begin{tikzcd}
b(a(t_v)) \arrow[r, "v"] \arrow[d] & b(a(p(\sigma))) \arrow[d] \\
b(\mathrm{id}_\sigma) \arrow[r, "b(d^s)"] \arrow[d, "="'] & b((\sigma^*)^s(\sigma)) \arrow[d] \\
b(\mathrm{id}_\sigma) \arrow[r, "d"] \arrow[d] & b(\sigma^*(\sigma)) \arrow[d] \\
t_v \arrow[r, "v"] & p(\sigma)
\end{tikzcd}$$

arising from the definitions of a, d^s and b. The homotopy $\phi(\sigma)$ is a composition of the three right faces. Thus a composition of the three squares with fixed composition $\phi(\sigma)$ of the right right faces is a homotopy ψ as required. $\square$

Proposition 4.85. *The covariant cwf* Coa sLcc *supports dependent product types along base types. That is, if $\Gamma \vdash \sigma$ is a base type and $\Gamma.\sigma \vdash \tau$ is an arbitrary type, then there is a type $\Gamma \vdash \Pi_\sigma \tau$ with term constructors*

$$\frac{\Gamma.\sigma \vdash t : \tau}{\Gamma \vdash \mathrm{lam}(t) : \Pi_\sigma \tau} \qquad \frac{\Gamma \vdash u : \Pi_\sigma \tau}{\Gamma.\sigma \vdash \mathrm{app}(u) : \tau}$$

$$\frac{\Gamma.\sigma \vdash t : \tau}{\Gamma.\sigma \vdash \mathrm{app}_\beta(t) : \mathrm{Id}\,(\mathrm{app}(\mathrm{lam}(t)))\, t}$$

$$\frac{\Gamma \vdash u_1 : \Pi_\sigma \tau \qquad \Gamma \vdash u_2 : \Pi_\sigma \tau \qquad \Gamma.\sigma \vdash h : \mathrm{Id}\,(\mathrm{app}(u_1))\,(\mathrm{app}(u_2))}{\Gamma \vdash \mathrm{funext}(h) : \mathrm{Id}\, u_1\, u_2}$$

$$\frac{\Gamma \vdash u : \Pi_\sigma \tau}{\Gamma \vdash \mathrm{funext}_\beta(u) : \mathrm{Id}\, \mathrm{funext}(\mathrm{refl}(\mathrm{app}(u)))\, \mathrm{refl}(u)}$$

Proof. Given a type $\Gamma.\sigma \vdash \tau$, we define $\Gamma \vdash \Pi_\sigma \tau$ by application of the functor

$$U(G(\Gamma.\sigma)) \xrightarrow{a} U(G(\Gamma^{/\sigma})) \xrightarrow{\Pi_{!_\sigma}} U(G(\Gamma^{/1})) \longrightarrow U(G(\Gamma)) \tag{4.43}$$

to τ, where $!_\sigma : \sigma \to 1$ is the canonical map to the terminal object in Γ. a is lcc and hence preserves terminal objects, Π_σ preserves them because it is a right adjoint, and $U(G(\Gamma^{/1})) \to U(G(\Gamma))$ is preserves terminal objects because it is a categorical equivalence. Thus if $\Gamma.\sigma \vdash t : \tau$, then the image of $1_t \xrightarrow{t} \tau$ under the functor (4.43) defines a term $\Gamma \vdash \mathrm{lam}(t) : \Pi_\sigma \tau$.

Now let $\Gamma \vdash u : \Pi_\sigma \tau$. Pulling back u along σ, we obtain a diagram

$$\begin{tikzcd}
& & & \cdot \arrow[rr] \arrow[dl] \arrow[dd, "\sigma^*(1_u)"] & & 1_u \arrow[dl, "u"'] \arrow[dd] \\
\cdot \arrow[dd, "a(\tau)"'] \arrow[ddrr, phantom, "\varepsilon"] & & \cdot \arrow[ll] \arrow[rr] \arrow[dd, "\sigma^*(\Pi_\sigma \tau)"] \arrow[dr, phantom, "\sigma^*(u)"] & & \Pi_\sigma \tau \arrow[dd] & \\
& & & \sigma \arrow[rr] \arrow[dl] & & 1 \arrow[dl] \\
\sigma & & \sigma \arrow[ll] \arrow[rr] & & 1 &
\end{tikzcd} \tag{4.44}$$

with the counit $\varepsilon = \varepsilon(a(\tau))$ of the adjunction $\sigma^* \vdash \Pi_\sigma$. Now $\varepsilon \circ \sigma^*(u) : \sigma^*(1_u) \to a(\tau)$ in $\Gamma^{/\sigma}$, and the composite

$$\mathrm{app}(u) : b(\sigma^*(1_u)) \xrightarrow{b(\varepsilon \circ \sigma^*(u))} b(a(\tau)) \xrightarrow{\alpha(\tau)} \tau$$

is a term $\Gamma.\sigma \vdash \mathrm{app}(u) : \tau$ as required.

Now let $\Gamma.\sigma \vdash t : \tau$ and let us construct $\Gamma.\sigma \vdash \mathrm{app}_\beta(t) : \mathrm{Id}\,(\mathrm{app}(\mathrm{lam}(t)))\,t$. Combining the definition of $\mathrm{lam}(t)$ with diagram (4.44), we obtain a diagram

$$\begin{tikzcd}
& \cdot \arrow[dl] \arrow[dd, "a(1_t)"] & & \cdot \arrow[ll] \arrow[rr] \arrow[dl] \arrow[dd, "\sigma^*(1_{\mathrm{lam}(t)})"] & & 1_{\mathrm{lam}(t)} \arrow[dl, "\mathrm{lam}(t)"'] \arrow[dd] \\
\cdot \arrow[dd, "a(\tau)"'] \arrow[dr, phantom, "a(t)"] & & \cdot \arrow[ll] \arrow[rr] \arrow[dd] & & \Pi_\sigma \tau \arrow[dd] & \\
& \sigma \arrow[dl] & & \sigma \arrow[ll] \arrow[rr] \arrow[dl] & & 1 \arrow[dl] \\
\sigma & & \sigma \arrow[ll] \arrow[rr] & & 1 &
\end{tikzcd}$$

From left to right, the squares along the y and z axes are given by t, by $\sigma^*(\mathrm{lam}(t))$ and by $\Pi_\sigma(a(t))$. The left cube is given by $\varepsilon(a(t))$ and witnesses an equivalence $h : \varepsilon(a(\tau)) \circ \sigma^* \simeq a(t)$ of maps in $\Gamma^{/\sigma}$ which restricts to the identity on the codomain. Then $b(h) \circ \alpha(\tau)$ is an equivalence $\mathrm{app}(\mathrm{lam}(t)) \simeq t$ in $\Gamma.\sigma$, which induces the required term $\Gamma.\sigma \vdash \mathrm{app}_\beta(t) : \mathrm{Id}\,(\mathrm{app}(\mathrm{lam}(t)))\,t$.

Next let $\Gamma \vdash u_0 : \Pi_\sigma \tau$, $\Gamma \vdash u_1 : \Pi_\sigma \tau$, $\Gamma.\sigma \vdash h : \mathrm{Id}\,\mathrm{app}(u_0)\,\mathrm{app}(u_1)$ and let us define the function extensionality term $\Gamma \vdash \mathrm{funext}(u) : \mathrm{Id}\,u_0\,u_1$. By the definition of the application terms of the diagram (4.44) and the homotopy inverse a to b (with natural equivalence $\beta : a \circ b \simeq \mathrm{id}$), we obtain from h a diagram

$$
\begin{tikzcd}
& & a(\tau) \arrow[dl, "="] \arrow[ddll, bend right, "="'] & & & \sigma^*(1_{u_0}) \arrow[lll, "\varepsilon\circ\sigma^*(u_0)"'] \arrow[dl] \arrow[ddll, bend right, "\sigma^*(u_0)"'] \\
& a(\tau) \arrow[dl, "="] & & & \sigma^*(1_{u_1}) \arrow[lll, "\varepsilon\circ\sigma^*(u_1)"] \arrow[dl, "\sigma^*(u_1)"] & \\
a(\tau) & & & \sigma^*(\Pi_\sigma \tau) \arrow[lll, "\varepsilon"] & &
\end{tikzcd}
\tag{4.45}
$$

in $\Gamma^{/\sigma}$. This diagram induces an equivalence $u_0 \simeq u_1$ via a lift against $\tilde{j}^1_\Pi$, hence a term $\Gamma \vdash \mathrm{funext}(h) : \mathrm{Id}\, u_0\, u_1$.

Finally, given $\Gamma \vdash u : \Pi_\sigma \tau$, let us construct the term $\Gamma \vdash \mathrm{funext}_\beta : \mathrm{Id}\,\mathrm{funext}(\mathrm{refl}(\mathrm{app}(u)))\,\mathrm{refl}(u)$. For $h = \mathrm{refl}(u)$, the diagram (4.45) is equivalent to a diagram in which the square with vertices $a(\tau)$ and $\sigma^*(1_{u_i})$ is degenerated. This degenerated diagram can be extended along $\tilde{j}^1_\Pi$ using the degenerated homotopy on $u = u_0 = u_1$. By lifting against the pushout product of $\partial\Delta^1 \subseteq (\Delta^1)^\sharp$ and $\tilde{j}^1_\Pi$, we obtain a homotopy of the homotopy $u_0 \simeq u_1$ as in the construction of $\mathrm{funext}(\mathrm{refl}(u))$ with the degenerated homotopy on u. This homotopy of homotopies induces the term $\Gamma \vdash \mathrm{funext}_\beta : \mathrm{Id}\,\mathrm{funext}(\mathrm{refl}(\mathrm{app}(u)))\,\mathrm{refl}(u)$. $\square$

Bibliography

[1] Thorsten Altenkirch, Conor McBride, and Wouter Swierstra. Observational equality, now! In *Proceedings of the 2007 Workshop on Programming Languages Meets Program Verification*, PLPV '07, page 57–68. ACM, 2007. ISBN 9781595936776. 30

[2] Thorsten Altenkirch, Nils Anders Danielsson, and Nicolai Kraus. Partiality, revisited - the partiality monad as a quotient inductive-inductive type. In *FOSSACS 2017*, pages 534–549, 2017. 28, 33, 34, 39

[3] Thorsten Altenkirch, Paolo Capriotti, Gabe Dijkstra, Nicolai Kraus, and Fredrik Nordvall Forsberg. Quotient inductive-inductive types. In *International Conference on Foundations of Software Science and Computation Structures*, pages 293–310. Springer, 2018. 28, 33, 34

[4] D. W. Anderson. Fibrations and geometric realizations. *Bulletin of the American Mathematical Society*, 84(5):765–788, 09 1978. 67

[5] Mathieu Anel. Computing enveloping ∞-topoi, 2020. URL `http://mathieu.anel.free.fr/mat/doc/Anel-2020-MURImeeting.pdf`. 31

[6] Philippe Audebaud and Christine Paulin-Mohring. Proofs of randomized algorithms in Coq. In *MPC*, 2006. 3, 27, 57

[7] Steve Awodey and Michael A Warren. Homotopy theoretic models of identity types. In *Mathematical proceedings of the cambridge philosophical society*, volume 146, pages 45–55. Cambridge University Press, 2009. 22

[8] Andrej Bauer, Jason Gross, Peter LeFanu Lumsdaine, Michael Shulman, Matthieu Sozeau, and Bas Spitters. The HoTT library: A formalization of homotopy type theory in coq. In *Proceedings of the 6th ACM SIGPLAN Conference on Certified Programs and Proofs*, CPP 2017, pages 164–172. ACM, 2017. ISBN 978-1-4503-4705-1. 28

[9] Santiago Zanella Béguelin. *Formal certification of game-based cryptographic proofs.* PhD thesis, Ecole Nationale des Mines de Paris, 2010. 27

[10] Martin E. Bidlingmaier. An interpretation of dependent type theory in a model category of locally cartesian closed categories, 2020. 98, 99, 139

[11] Martin E. Bidlingmaier, Florian Faissole, and Bas Spitters. Synthetic topology in homotopy type theory for probabilistic programming. *Mathematical Structures in Computer Science*, page 1–29, 2021. doi: 10.1017/S0960129521000165. 1

[12] Robert Blackwell, Gregory Maxwell Kelly, and A. John Power. Two-dimensional monad theory. *Journal of pure and applied algebra*, 59(1): 1–41, 1989. 83

[13] Rafaël Bocquet. Coherence of strict equalities in dependent type theories. *arXiv preprint arXiv:2010.14166*, 2020. 24

[14] John Bourke. Equipping weak equivalences with algebraic structure. *Mathematische Zeitschrift*, Apr 2019. ISSN 1432-1823. doi: 10.1007/s00209-019-02305-w. 82, 143, 144

[15] John Bourke and Richard Garner. Algebraic weak factorisation systems i: Accessible awfs. *Journal of Pure and Applied Algebra*, 220(1):108–147, Jan 2016. ISSN 0022-4049. doi: 10.1016/j.jpaa.2015.06.002. 138

[16] Douglas Bridges and Fred Richman. *Varieties of constructive mathematics*, volume 97. Cambridge University Press, 1987. 38

[17] Michael Ching and Emily Riehl. Coalgebraic models for combinatorial model categories. *Homology, Homotopy and Applications*, 16(2):171–184, 2014. 67, 86

[18] Pierre Clairambault and Peter Dybjer. The biequivalence of locally cartesian closed categories and martin-löf type theories. In *International Conference on Typed Lambda Calculi and Applications*, pages 91–106. Springer, 2011. 95

[19] Thierry Coquand. *An analysis of Girard's paradox*. PhD thesis, INRIA, 1986. 18

[20] Thierry Coquand. Constructive mathematics and higher topos theory, 2019. URL `http://www.cse.chalmers.se/~coquand/birmingham.pdf`. 61

[21] Thierry Coquand and Erik Palmgren. Metric boolean algebras and constructive measure theory. *Archive for Mathematical Logic*, 41(7):687–704, 2002. 43

[22] Thierry Coquand and Bas Spitters. Integrals and valuations. *Journal of Logic and Analysis*, 1(3):1–22, 2009. ISSN 1759-9008. 43, 46

[23] Thierry Coquand, Bassel Mannaa, and Fabian Ruch. Stack semantics of type theory. In *2017 32nd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 1–11. IEEE, 2017. 61

[24] Pierre-Louis Curien. Substitution up to isomorphism. *Diagrammes*, 23: 43–66, 1990. 19, 64

[25] Pierre-Louis Curien, Richard Garner, and Martin Hofmann. Revisiting the categorical interpretation of dependent type theory. *Theoretical Computer Science*, 546:99–119, 2014. ISSN 0304-3975. doi: https://doi.org/10.1016/j.tcs.2014.03.003. URL `https://www.sciencedirect.com/science/article/pii/S0304397514001789`. Models of Interaction: Essays in Honour of Glynn Winskel. 19

[26] Eduardo J. Dubuc. *Kan extensions in enriched category theory*, volume 145. Springer, 1970. 83

[27] Peter Dybjer. Internal type theory. In *International Workshop on Types for Proofs and Programs*, pages 120–134. Springer, 1995. 8, 64, 80

[28] Martín Escardó. Synthetic topology: of data types and classical spaces. *ENTCS*, 87:21–156, 2004. 36, 61

[29] Martin Escardó and Chuangjie Xu. A constructive manifestation of the kleene–kreisel continuous functionals. *Annals of Pure and Applied Logic*, 167(9):770–793, 2016. 61

[30] Martín H. Escardó and Cory M. Knapp. Partial Elements and Recursion via Dominances in Univalent Type Theory. In Valentin Goranko and Mads Dam, editors, *CSL 2017*, volume 82 of *LIPIcs*, pages 21:1–21:16. Schloss Dagstuhl, 2017. ISBN 978-3-95977-045-3. 38, 39, 60

[31] Florian Faissole and Bas Spitters. Preuves constructives de programmes probabilistes. In *Presentation at JFLA 2018 - Journées Francophones des Langages Applicatifs*, 2018. URL `https://hal.inria.fr/hal-01654459`. 29

[32] Michael Fourman. Continuous truth I: Non-constructive objects. In *Studies in Logic and the Foundations of Mathematics*, volume 112, pages 161–180. Elsevier, 1984. 31

[33] Michael Fourman. Continuous truth II: Reflections. In *WoLLIC*, 2013. 31

[34] Geätan Gilbert. Formalising real numbers in homotopy type theory. *CPP'17*, 2016. 28, 37, 39, 40, 59

[35] Michele Giry. A categorical approach to probability theory. In *Categorical aspects of topology and analysis*, pages 68–85. Springer, 1982. 27

[36] Bertrand Guillou and J. Peter May. Enriched model categories and presheaf categories, 2011. 67

[37] Kathryn Hess, Magdalena Kędziorek, Emily Riehl, and Brooke Shipley. A necessary and sufficient condition for induced model structures. *Journal of Topology*, 10(2):324–369, Apr 2017. ISSN 1753-8424. doi: 10.1112/topo.12011. URL `http://dx.doi.org/10.1112/topo.12011`. 143

[38] C. Heunen, O. Kammar, S. Staton, and H. Yang. A convenient category for higher-order probability theory. In *2017 32nd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 1–12, 2017. 4, 61

[39] Philip S. Hirschhorn. *Model categories and their localizations*, volume 99. American Mathematical Soc., 2009. 67, 73, 76, 146

[40] Martin Hofmann. On the interpretation of type theory in locally cartesian closed categories. In *International Workshop on Computer Science Logic*, pages 427–441. Springer, 1994. 19, 64, 67

[41] Martin Hofmann and Thomas Streicher. The groupoid interpretation of type theory. *Twenty-five years of constructive type theory (Venice, 1995)*, 36:83–111, 1998. 21

[42] Alfred Horn and Alfred Tarski. Measures in Boolean algebras. *Transactions of the American Mathematical Society*, 64(3):467–497, 1948. 46

[43] Mark Hovey. *Model categories*. Number 63. American Mathematical Soc., 2007. 19

[44] Daniel Huang. *On Programming Languages for Probabilistic Modeling*. PhD thesis, Harvard, 2017. 29

[45] Daniel Huang, Greg Morrisett, and Bas Spitters. An application of computable distributions to the semantics of probabilistic programs. In Gilles Barthe, Alexandra Silva, and Joost-Pieter Katoen, editors, *Foundations of Probabilistic Programming*, 2020. 29

[46] Martin Hyland. First steps in synthetic domain theory. In *Category Theory*, pages 131–156. Springer, 1991. 36

[47] Valery Isaev. Model category of marked objects, 2016. URL `https://arxiv.org/abs/1610.08459`. 20, 67, 68, 69, 70, 101, 102, 110

[48] B. Jacobs. *Categorical Logic and Type Theory*. Number 141 in Studies in Logic and the Foundations of Mathematics. North Holland, Amsterdam, 1999. 169

[49] Peter T Johnstone. On a topological topos. *Proceedings of the London mathematical society*, 3(2):237–271, 1979. 61

[50] Peter T Johnstone. *Sketches of an elephant: A topos theory compendium*, volume 2. Oxford University Press, 2002. 31, 40

[51] Claire Jones and Gordon Plotkin. A probabilistic powerdomain of evaluations. In *LICS*, 1989. 43

[52] André Joyal. Notes on quasi-categories. *preprint*, 2008. 114

[53] Achim Jung, M Andrew Moshier, and Steve Vickers. Presenting dcpos and dcpo algebras. *Electronic Notes in Theoretical Computer Science*, 218: 209–229, 2008. 32, 33, 35

[54] Krzysztof Kapulkin. Locally cartesian closed quasi-categories from type theory. *Journal of Topology*, 10(4):1029–1049, Nov 2017. ISSN 1753-8424. doi: 10.1112/topo.12031. URL `http://dx.doi.org/10.1112/topo.12031`. 23, 98

[55] Krzysztof Kapulkin and Peter LeFanu Lumsdaine. The simplicial model of univalent foundations (after voevodsky). *Journal of the European Mathematical Society*, 23(6):2071–2126, 2021. 22

[56] Krzysztof Kapulkin and Karol Szumiło. Internal languages of finitely complete $(\infty, 1)$-categories. *Selecta Mathematica*, 25(2):33, 2019. 23, 98

[57] Gregory Maxwell Kelly and Max Kelly. *Basic concepts of enriched category theory*, volume 64. CUP Archive, 1982. 71, 83

[58] Stephen Lack. Homotopy-theoretic aspects of 2-monads. *Journal of Homotopy and Related Structures*, 2(2):229–260, 2007. 67, 84

[59] Davorin Lešnik. Synthetic topology and constructive metric spaces. *Diss. University of Ljubljana*, 2010. 5, 28, 31, 36, 40, 52, 54, 55

[60] Peter Lumsdaine and Michael Shulman. Semantics of higher inductive types. *Mathematical Proceedings of the Cambridge Philosophical Society*, pages 1–50, 2017. 34

[61] Peter LeFanu Lumsdaine and Michael A Warren. The local universes model: an overlooked coherence construction for dependent type theories. *ACM Transactions on Computational Logic (TOCL)*, 16(3):1–31, 2015. 19, 64, 67

[62] Jacob Lurie. *Higher topos theory.* Princeton University Press, 2009. 85, 102, 104, 105, 106, 114, 124, 125, 134, 135

[63] Maria Emilia Maietti. Joyal's arithmetic universe as list-arithmetic pretopos. *Theory & Applications of Categories*, 24, 2010. 95

[64] G. Marsaglia and T. A. Bray. A convenient method for generating normal variables. *SIAM Review*, 6(3):260–264, 1964. 60

[65] Eugenio Moggi. Notions of computation and monads. *Information and computation*, 93(1):55–92, 1991. 3, 27, 50

[66] Thomas Nikolaus. Algebraic models for higher categories. *Indagationes Mathematicae*, 21(1-2):52–75, 2011. 67, 77, 82, 83, 138, 143

[67] E. Palmgren and S.J. Vickers. Partial horn logic and cartesian categories. *Annals of Pure and Applied Logic*, 145(3):314–353, 2007. ISSN 0168-0072. doi: https://doi.org/10.1016/j.apal.2006.10.001. URL `https://www.sciencedirect.com/science/article/pii/S0168007206001229`. 6

[68] Gordon Plotkin and John Power. Adequacy for algebraic effects. In *International Conference on Foundations of Software Science and Computation Structures*, pages 1–24. Springer, 2001. 57, 58, 59

[69] Daniel G. Quillen. *Homotopical Algebra*. Springer Berlin Heidelberg, 1967. doi: 10.1007/bfb0097438. URL `https://doi.org/10.1007/bfb0097438`. 19

[70] Charles Rezk. Stuff about quasicategories, 2021. URL `https://faculty.math.illinois.edu/~rezk/quasicats.pdf`. 107, 130

[71] Emily Riehl. *Categorical Homotopy Theory*. New Mathematical Monographs. Cambridge University Press, 2014. doi: 10.1017/CBO9781107261457. 139, 145

[72] Emily Riehl and Dominic Verity. The 2-category theory of quasi-categories. *Advances in Mathematics*, 280:549–642, 2015. 106

[73] Egbert Rijke and Bas Spitters. Sets in homotopy type theory. In *MSCS, special issue: From type theory and homotopy theory to univalent foundations*, 2014. 29, 31

[74] G. Rosolini. *Continuity and effectiveness in topoi*. PhD thesis, University of Oxford, 1986. 34, 37, 38

[75] Robert A. G. Seely. Locally cartesian closed categories and type theory. In *Mathematical proceedings of the Cambridge philosophical society*, volume 95, pages 33–48. Cambridge University Press, 1984. 13, 63

[76] Michael Shulman. Brouwer's fixed-point theorem in real-cohesive homotopy type theory. *Mathematical Structures in Computer Science*, 28(6):856–941, 2018. 61

[77] Michael Shulman. All $(\infty, 1)$-toposes have strict univalent universes, 2019. 23, 31

[78] Bas Spitters and Elis Van der Weegen. Type classes for mathematics in type theory. *MSCS, special issue on 'Interactive theorem proving and the formalization of mathematics'*, 21:1–31, 2011. 28

[79] Sam Staton, Hongseok Yang, Chris Heunen, Ohad Kammar, and Frank Wood. Semantics for probabilistic programming: higher-order functions, continuous distributions, and soft constraints. *CoRR*, abs/1601.04943, 2016. 61

[80] Jonathan Sterling, Carlo Angiuli, and Daniel Gratzer. A cubical language for bishop sets. *arXiv preprint arXiv:2003.01491*, 2020. 30

[81] Thomas Streicher. Universes in toposes. *From Sets and Types to Topology and Analysis, Towards Practicable Foundations for Constructive Mathematics*, 48:78–90, 2005. 17

[82] Andrew Swan and Taichi Uemura. On Church's thesis in cubical assemblies. *arXiv preprint arXiv:1905.03014*, 2019. 23, 31

[83] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations for Mathematics.* `http://homotopytypetheory.org/book`, Institute for Advanced Study, 2013. 22, 28, 33

[84] Matthijs Vákár, Ohad Kammar, and Sam Staton. A domain theory for statistical probabilistic programming. *Proc. ACM Program. Lang.*, 3(POPL), jan 2019. doi: 10.1145/3290349. URL `https://doi.org/10.1145/3290349`. 4

[85] Benno van den Berg and Martijn den Besten. Quadratic type checking for objective type theory, 2021. 24

[86] Gerrit Van Der Hoeven and Ieke Moerdijk. On choice sequences determined by spreads. *The Journal of symbolic logic*, 49(3):908–916, 1984. 31

[87] Jaap Van Oosten. *Realizability: an introduction to its categorical side.* Elsevier, 2008. 5

[88] Steven Vickers. Locales and toposes as spaces. In *Handbook of spatial logics*, pages 429–496. Springer, 2007. 65

[89] Steven Vickers. Locales and toposes as spaces. In Marco Aiello, Ian Pratt-Hartmann, and Johan Van Benthem, editors, *Handbook of spatial logics*, pages 429–496. Springer, 2007. 17

[90] Steven Vickers. A localic theory of lower and upper integrals. *Mathematical Logic Quarterly*, 54(1):109–123, 2008. 61

[91] Steven Vickers. A monad of valuation locales. 2011. URL `http://www.cs.bham.ac.uk/sjv/Riesz.pdf`. 43, 46, 51, 52, 61

[92] Steven Vickers. Sketches for arithmetic universes. *Journal of Logic and Analysis*, 08 2016. doi: 10.4115/jla.2019.11.FT4. 65, 95